Limits of Projective Manifolds Under Holomorphic Deformations

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Abstract. We prove that if in a complex analytic family of compact complex manifolds all the fibres, except one, are supposed to be projective, then the remaining (limit) fibre must be Moishezon. The proof is based on the so-called singular Morse inequalities for integral cohomology classes that we obtained in a previous work. The strategy, originating in the work of J.-P. Demailly, consists in using the Aubin-Calabi-Yau theorem to construct Kähler forms on non-limit fibres in a certain integral De Rham cohomology 2-class and in showing that this family of forms is bounded in mass in a suitable sense. By weak compactness, a subsequence of Kähler forms converges weakly to a $(1,1)$-current that may have wild singularities and is defined on the limit fibre. The singular Morse inequalities are then used on the limit fibre to produce a Kähler current in the same integral cohomology class. The existence of a Kähler current with integral cohomology class is known to characterise Moishezon manifolds.

1 Introduction

A complex analytic family of compact complex manifolds is a proper holomorphic submersion $\pi : \mathcal{X} \to \Delta$ between complex manifolds $\mathcal{X}$ and Δ ([Kod86]). Thus all the fibres are (smooth) compact complex manifolds of equal dimensions. The base manifold Δ will be assumed to be an open ball containing the origin in some complex space $\mathbb{C}^m$. The purpose of this paper is to prove the following statement.

Theorem 1.1 *Let $\pi : \mathcal{X} \longrightarrow \Delta$ be a complex analytic family of compact complex manifolds such that the fibre $X_t := \pi^{-1}(t)$ is projective for every $t \in \Delta^\star := \Delta \setminus \{0\}$. Then $X_0 := \pi^{-1}(0)$ is Moishezon.*

Recall that a compact complex manifold X is said to be Moishezon if there exists a proper holomorphic bimeromorphic map (i.e. a holomorphic modification) $\mu : \tilde{X} \to X$ such that $\tilde{X}$ is a projective manifold. This condition is equivalent to the existence of n algebraically independent meromorphic functions on X where $n = \dim_{\mathbb{C}} X$ ([Moi67]). A Moishezon manifold becomes projective after finitely many blow-ups with smooth centres ([Moi67]). Thus Theorem 1.1 says that projective manifolds can degenerate only mildly (i.e.

to Moishezon manifolds) in the deformation limit. Note that the result is optimal since, by Hironaka's example [Hir62], the limit fibre X_0 need not be Kähler, let alone projective. (*A posteriori*, since X_0 is Moishezon by Theorem 1.1, X_0 cannot be Kähler unless it is projective — see [Moi67]).

As is well known in deformation theory ([Kod86]), all the fibres $X_t := \pi^{-1}(t)$ are C^∞-diffeomorphic to a fixed compact differentiable manifold X. In other words, the family of complex manifolds $(X_t)_{t\in\Delta}$ can be seen as one differentiable manifold X equipped with a family of complex structures $(J_t)_{t\in\Delta}$ varying in a holomorphic way with t. In particular, for every k, the De Rham cohomology groups $H^k(X_t,\, \mathbb{C})$ of all the fibres can be identified with a fixed $H^k(X,\, \mathbb{C})$, while the Dolbeault cohomology groups $H^{p,\,q}(X_t,\, \mathbb{C})$ depend on $t \in \Delta$.

As the fibres X_t are assumed to be projective for $t \neq 0$, the following fact is classical.

Remark 1.2 *There exists a non-zero integral De Rham cohomology* 2*-class* $\alpha \in H^2(X,\, \mathbb{Z})$ *such that, for every* $t \in \Delta^\star$, α *can be represented by a* 2*-form which is of* J_t*-type* $(1,1)$.

Moreover, α *can be chosen in such a way that, for every* $t \in \Delta\setminus\Sigma$, α *is the first Chern class of an ample line bundle* $L_t \to X_t$, *where* $\Sigma = \{0\} \cup \Sigma' \subset \Delta$ *and* $\Sigma' = \bigcup \Sigma_\nu$ *is a countable union of proper analytic subsets* $\Sigma_\nu \subsetneq \Delta^\star$.

To see this well-known fact, for any given class $\alpha \in H^2(X,\mathbb{R})$, let $S_\alpha \subset \Delta^\star$ denote the set of points $t \in \Delta^\star$ such that α can be represented by a J_t-type $(1,1)$-form. For every $t \in \Delta^\star$, X_t is compact Kähler (even projective), so there exists a Hodge decomposition $H^2(X,\mathbb{C}) = H^{2,0}(X_t,\mathbb{C}) \oplus H^{1,1}(X_t,\mathbb{C}) \oplus H^{0,2}(X_t,\mathbb{C})$ with $H^{2,0}(X_t,\mathbb{C}) = \overline{H^{0,2}(X_t,\mathbb{C})}$. Thus, a given $\alpha \in H^2(X,\mathbb{R})$ contains a J_t-type $(1,1)$-form if and only if its projection onto $H^{0,2}(X_t,\mathbb{C})$ vanishes. This means that S_α is the set of zeroes of the section $\sigma_\alpha \in \Gamma(\Delta^\star,\, R^2\pi_\star\mathcal{O}_\mathcal{X})$ induced by α. By the Kähler assumption on every X_t with $t \neq 0$, the map $\Delta^\star \ni t \mapsto \dim H^{0,2}(X_t,\mathbb{C})$ is locally constant and therefore the restriction of the higher direct image sheaf $R^2\pi_\star\mathcal{O}_\mathcal{X}$ to $\Delta^\star$ is locally free. As J_t varies holomorphically with t, σ_α is a holomorphic section of the associated holomorphic vector bundle over $\Delta^\star$. This clearly implies that S_α is an analytic subset of $\Delta^\star$ for every $\alpha \in H^2(X,\mathbb{R})$. On the other hand, the projectiveness assumption on every X_t with $t \neq 0$ entails the equality

$$\bigcup_\alpha S_\alpha = \Delta^\star, \tag{1}$$

where the union is taken over all the integral classes $\alpha \in H^2(X,\mathbb{Z})$ such that α is an ample class on some fibre X_{t_0}, $t_0 \neq 0$ (depending on α). Now, a proper analytic subset is Lebesgue negligible. If S_α were a proper subset of $\Delta^\star$ for every such $\alpha \in H^2(X,\mathbb{Z})$, the left-hand side in (1) would be a countable union of Lebesgue negligible subsets, hence a Lebesgue negligible

subset of $\Delta^\star$, contradicting the equality to $\Delta^\star$. Therefore, there must exist $\alpha \in H^2(X,\mathbb{Z})$ which can be represented by a J_t-type $(1,1)$-form for every $t \neq 0$ (i.e. $S_\alpha = \Delta^\star$) and which is an ample class on at least one fibre X_{t_0}, $t_0 = t_0(\alpha) \neq 0$.

Now, it is a standard fact that the ampleness property is open with respect to the countable analytic Zariski topology of the punctured base $\Delta^\star$ (over which the fibres are projective). This follows from the Nakai-Moishezon criterion (according to which ampleness can be tested as numerical strict positivity on all classes of analytic cycles of the given projective manifold X_t, $t \neq 0$) and Barlet's theory of cycle spaces ([Bar75]) which implies that the cohomology classes $\{[Z]\}$ of analytic cycles $Z \subset X_t$ with $t \neq 0$ are the same on all fibres X_t, $t \neq 0$, except possibly on a countable union of analytic subsets of exceptional fibres which may have more classes of cycles than the generic fibre (see e.g. [DP04, § 5.] where the argument is extended to Kähler fibres using the transcendental version of the Nakai-Moishezon criterion obtained as the main result of that work).

Hence, as α is an ample class on some fibre X_{t_0} with $t_0 \neq 0$, α must be an ample class on every fibre X_t with $t \in \Delta \setminus \Sigma$, where $\Sigma = \{0\} \cup \Sigma'$ for a countable union $\Sigma' = \bigcup \Sigma_\nu$ of proper analytic subsets $\Sigma_\nu \subsetneq \Delta^\star$. $\square$

Let $n := \dim_{\mathbb{C}} X_t$, $t \in \Delta$. Fix a class $\alpha \in H^2(X,\, \mathbb{Z})$ as above and set

$$v := \int\limits_X \alpha^n > 0. \tag{2}$$

By Stokes' theorem, this integral is clearly independent of the choice of representative of α. Moreover, $v > 0$ since α is the first Chern class of an ample line bundle L_t on X_t and $v = L_t^n > 0$ is the volume of L_t for every $t \in \Delta \setminus \Sigma$. Finally, the differential operator d of X admits a separate splitting

$$d = \partial_t + \bar\partial_t, \qquad t \in \Delta,$$

for each complex structure J_t of X.

The proof of Theorem 1.1 will evolve from a strategy devised in broad outline and propounded over the years by J.-P. Demailly aiming at producing a Kähler current on the limit fibre X_0. Recall that a d-closed (1, 1)-current T is said to be a *Kähler current* (a term coined in [JS93]) if $T \geq \varepsilon\,\omega$ for some $\varepsilon > 0$ and some positive-definite C^∞ Hermitian (1, 1)-form $\omega > 0$ on the ambient manifold. This is a strong notion of strict positivity for currents. Within the class of compact complex manifolds, the existence of a Kähler current characterises *Fujiki class* $\mathcal{C}$ *manifolds* (i.e. those admitting a holomorphic modification to a compact Kähler manifold, much as Moishezon manifolds modify to projective ones) by a result of [DP04], while the existence of a Kähler current with integral De Rham cohomology class characterises Moishezon manifolds ([JS93], see also [Dem90]). Thus, the pair Fujiki

class $\mathcal{C}$/Moishezon bears a striking similarity to the pair Kähler/projective : by Kodaira's Embedding Theorem, projective manifolds are precisely those compact complex manifolds carrying a Kähler metric with integral De Rham cohomology class. The former pair can be seen as the *current* version of the latter, while the latter term in each pair is the *integral class* version of the former.

The thrust of Demailly's Morse inequalities ([Dem85] and further developments) is to produce a Kähler current in a given cohomology class when the class satisfies comparatively weak positivity properties. This idea had motivated our previous work [Pop08] which is to be made a crucial use of in the present paper.

Here is an outline of our approach. Consider a C^∞ family $(dV_t)_{t\in\Delta}$ of C^∞ volume forms $dV_t>0$ on X_t normalised such that $\int\limits_{X_t} dV_t = 1$. We can apply the Aubin-Yau theorem ([Yau78]) on the Calabi conjecture to the class α viewed as a Kähler class on X_t for every $t\in\Delta\setminus\Sigma$. Thus, for $t\in\Delta\setminus\Sigma$, we get a C^∞ 2-form $\omega_t\in\alpha=c_1(L_t)$ on X which is a Kähler form with respect to the complex structure J_t (i.e. $d\omega_t=0$, ω_t is of type $(1,\,1)$ and positive definite with respect to J_t) such that

$$\omega_t^n(x) = v\,dV_t(x), \qquad x\in X_t. \tag{3}$$

The first step in the proof of Theorem 1.1 will be to show that the family of Kähler forms $(\omega_t)_{t\in\Delta\setminus\Sigma}$ is bounded in mass (in a suitable sense that will be made precise below) as t approaches 0. By weak compactness, this family will contain a subsequence that is weakly convergent to a current T. The limit current T must be of J_0-type $(1,\,1)$ and must lie in the given class $\alpha\in H^2(X,\,\mathbb{Z})$. This current, possibly with wild singularities, will only satisfy mild positivity properties on X_0. However, the so-called singular Morse inequalities for integral classes that we obtained in [Pop08] will imply the existence of a Kähler current on X_0 lying in the same cohomology class α as T. The class being integral, this is equivalent to α being the first Chern class of a big line bundle over X_0 which, in turn, amounts to X_0 being Moishezon.

The second step in the proof of Theorem 1.1 will thus consist in a crucial application of the following theorem which was the main result in [Pop08]. Given an arbitrary compact complex manifold X with $\dim_{\mathbb{C}}X=n$, recall that the volume of a holomorphic line bundle $L\rightarrow X$, a birational invariant measuring the asymptotic growth of spaces of global holomorphic sections of high tensor powers of L, is standardly defined as

$$v(L) := \limsup_{k\rightarrow+\infty}\frac{n!}{k^n}\,h^0(X,\,L^k). \tag{4}$$

If L is ample, the volume is known to be given by $v(L)=\int\limits_X c_1(L)^n := L^n$, motivating notation (2). Theorem 1.3. in [Pop08] gives the following metric

characterisation of the volume :

$$v(L) = \sup \int_X T_{ac}^n, \tag{5}$$

where the supremum is taken over all positive currents $T \geq 0$ in the first Chern class of L and T_{ac} stands for the absolutely continuous part of T in the Lebesgue decomposition of its measure coefficients. The interesting inequality in (5) is "$\geq$" (*singular Morse inequalities*). Now, the following three facts are well known : $v(L) > 0$ if and only if the line bundle L is big, by definition of bigness. A line bundle $L \to X$ is big if and only if it can be equipped with a (possibly singular) Hermitian metric h whose curvature current $T := i\Theta_h(L)$ is > 0 on X (i.e. a Kähler current), by [Dem90] for a projective X and [JS93] for a general X. A compact complex manifold carries a big line bundle if and only if it is Moishezon, by [Moi67]. As any d-closed current of type $(1, 1)$ whose De Rham cohomology class is integral is always the curvature current of some holomorphic line bundle equipped with a (possibly singular) Hermitian metric, an equivalent way of formulating the "$\geq$" part of (5) above is the following.

Theorem 1.3 *(Rewording of Theorem 1.3. in [Pop08]) Let X be a compact complex manifold, $\dim_{\mathbb{C}} X = n$. If there exists a d-closed $(1, 1)$-current T on X whose De Rham cohomology class is integral and which satisfies*

$$(i)\ \ T \geq 0 \ \ on\ X; \qquad (ii)\ \ \int_X T_{ac}^n > 0,$$

then the cohomology class of T contains a Kähler current S. Implicitly, X is Moishezon.

This is the form in which we will use the result of [Pop08] on X_0. The limit current T obtained on the limit fibre X_0 at the end of the first step in the proof of Theorem 1.1 will be shown to satisfy the mild positivity conditions (i) and (ii) of Theorem 1.3 on X_0. By that theorem, the integral class α of T must contain a Kähler current, proving that X_0 is Moishezon.

We shall first prove Theorem 1.1 under the extra assumption that the Hodge number $h^{0,1}(t) := \dim H^{0,1}(X_t, \mathbb{C})$ is independent of $t \in \Delta$. This assumption enables one to uniformly bound the masses of the Kähler forms $(\omega_t)_{t\in\Delta\setminus\Sigma}$ with respect to a family of Gauduchon metrics on the fibres X_t varying in a C^∞ way with the parameter $t \in \Delta$. This is because the invariance of $h^{0,1}(t)$ amounts to the existence of a uniform positive lower bound for the smallest positive eigenvalue of the anti-holomorphic Laplacian Δ''_t as t varies in a neighbourhood of 0 in Δ. Hence, the inverses of these small positive eigenvalues are uniformly bounded above and so are the masses of the Kähler forms $(\omega_t)_{t\in\Delta\setminus\Sigma}$. This will occupy Section 2.

A Moishezon manifold is well known to admit a Hodge decomposition and to have its Hodge-Frölicher spectral sequence degenerate at $E_1^{\bullet}$. This implies that, once Theorem 1.1 has been proved, all the Hodge numbers $h^{p,\,q}(t) := \dim H^{p,\,q}(X_t,\,\mathbb{C})$, $p, q = 0, \dots, n$, will be locally constant as t varies in Δ. In particular, the situation considered in Section 2 is *a posteriori* seen to always occur. Section 2 implicitly shows that, if only $h^{0,\,1}(t)$ is assumed to not depend on t, all the $h^{p,\,q}(t)$ are independent of t.

It is worth noticing that Section 2 also proves the special case of Theorem 1.1 where all the fibres X_t are assumed to be compact complex surfaces in the following strengthened form that fails for higher dimensional fibres.

Proposition 1.4 *Let $\pi : \mathcal{X} \longrightarrow \Delta$ be a complex analytic family of compact complex* **surfaces** *such that the fibre $X_t := \pi^{-1}(t)$ is projective for every $t \in \Delta^\star := \Delta \setminus \{0\}$. Then $X_0 := \pi^{-1}(0)$ is projective.*

Indeed, the Hodge-Frölicher spectral sequence of any compact complex surface is known to degenerate at $E_1^{\bullet}$. Consequently, all the Hodge numbers $h^{p,q}(t)$ are locally constant in a family of surfaces. In particular, the situation considered in Section 2 occurs and, by the arguments given there, X_0 is Moishezon. On the other hand, the Betti numbers b_k of the fibres being always constant, the first Betti number b_1 of X_0 must be even. Now, by Kodaira's theory of classification of surfaces and Siu's result [Siu83] (see also [Buc99], [Lam99]), every compact complex surface with b_1 even is Kähler. The limit surface X_0 being both Moishezon and Kähler, it must be projective ([Moi67]).

Furthermore, the singular Morse inequalities are quite easy to prove on complex surfaces in a tremendously simpler way than the higher-dimensional, (possibly) non-Kähler case treated in [Pop08] : the regularisation theorem with mass control that we obtained there for currents follows easily on a compact complex surface by using Demailly's regularisation of currents and choosing a Gauduchon metric (on a complex surface, this is a Hermitian metric ω such that $\partial\bar\partial\omega = 0$). This choice ensures the boundedness of the Monge-Ampère masses of Demailly's regularising currents (all of which lie in the same Bott-Chern cohomology class) thanks to Stokes' theorem, much as they are bounded on compact Kähler manifolds (the case treated in [Bou02]). In the general non-Kähler higher-dimensional case, a new regularisation had to be constructed in [Pop08] and the Monge-Ampère masses need not be bounded.

Thus, the case of families of compact complex surfaces is on a distinctly lower level of difficulty and interest than the general case. The main focus of this work will therefore be on families with fibre dimension ≥ 3.

Rather than proving the invariance of $h^{0,1}(t)$ on *a priori* grounds (a tall order that falls largely beyond the scope of this paper), we will prove Theorem 1.1 in full generality by working directly on Gauduchon metrics and the spectra of the associated Laplace operators. The method yields the desired

uniform mass boundedness of the family of Kähler forms $(\omega_t)_{t\in\Delta\setminus\Sigma}$ even in the mythical case where $h^{0,1}(t)$ jumps at $t=0$. Explicitly, we prove the following fact that can be regarded as the main technical result of this work.

Proposition 1.5 *Under the hypotheses of Theorem 1.1 and after possibly shrinking Δ about 0, there exists a family $(\gamma_t)_{t\in\Delta}$ of Gauduchon metrics varying in a C^∞ way with t on the fibres $(X_t)_{t\in\Delta}$ and satisfying the following uniform mass boundedness property. For every $t\in\Delta\setminus\Sigma$, choose any J_t-Kähler form ω_t belonging to the class $\alpha\in H^2(X,\mathbb{Z})$ given by Remark 1.2. Then there exists a constant $C>0$ independent of $t\in\Delta\setminus\Sigma$ such that*

$$0<\int\limits_{X_t}\omega_t\wedge\gamma_t^{n-1}\leq C<+\infty, \quad \textit{for all} \quad t\in\Delta\setminus\Sigma. \tag{6}$$

If $h^{0,1}(t)$ is independent of t near $0\in\Delta$, any choice of a smooth family of Gauduchon metrics will do (cf. Proposition 2.2). In general, a special family has to be constructed (cf. Propositions 3.4 and 4.1 which, between them, prove Proposition 1.5). The first moves will be made in Section 3 where a new kind of metric, strengthening Gauduchon metrics, is introduced. We call it a *strongly Gauduchon* metric and give an intrinsic necessary and sufficient condition for the existence of such a metric on an arbitrary compact complex manifold in terms of non-existence of certain $(1,1)$-currents. The method is the one of Sullivan [Sul76] that has been used for similar purposes in [HL83], [Mic83], [Lam99], [Buc99]. The conclusion of Section 3 will be a proof of another special case of Theorem 1.1 under the extra assumption that a *strongly Gauduchon* metric exists on the limit fibre X_0 (or, equivalently, that certain exceptional currents do not exist on X_0). This assumption is of a different nature to the one made in Section 2.

The proof of Theorem 1.1 will be obtained in full generality in Section 4 by reducing it to the case of Section 3 : the limit fibre X_0 will be shown to always carry a *strongly Gauduchon* metric if the other fibres are Kähler (or even more generally, if the $\partial\bar\partial$-lemma holds on the other fibres).

This naturally throws up new ideas to mount an attack on the following long-considered problem which will have by now become a matter of folklore.

Question 1.6 *Let $\pi:\mathcal{X}\longrightarrow\Delta$ be a complex analytic family of compact complex manifolds such that the fibre $X_t:=\pi^{-1}(t)$ is Kähler for every $t\in\Delta^\star:=\Delta\setminus\{0\}$. Then, is $X_0:=\pi^{-1}(0)$ a Fujiki class $\mathcal{C}$ manifold?*

Our Theorem 1.1 provides an affirmative answer to what can be seen as the *integral class* version of this question. The real class analogue of Remark 1.2 no longer holds in the more general context of Question 1.6 as there are examples of families with Kähler fibres for which no non-zero real De

Rham cohomology 2-class which is of type $(1,1)$ for all the complex structures involved exists. Thus, the constant class α has to be replaced with a C^∞ family of real classes $\alpha_t \in H^2(X,\mathbb{R})$, $t \neq 0$, whose volumes v_t remain uniformly bounded below away from zero near the origin in $\Delta^\star$. This can be arranged by standard arguments. Now, the significant fact is that our Proposition 1.5 still holds in this more general context if a suitable family of Kähler classes α_t, $t \neq 0$, replaces the constant class α. This is because the projective assumption on X_t with $t \neq 0$ is not made full use of in the proof of Proposition 1.5, but only the $\partial\bar\partial$-lemma and the Kähler assumption are used. This means that the only hurdle that has yet to be cleared before an (affirmative ?) answer to Question 1.6 can be given is Demailly's conjecture on *transcendental Morse inequalities* : the singular Morse inequalities that we obtained in [Pop08] and listed above as Theorem 1.3 are expected to hold without the *integral class* assumption on T (hence X would be Fujiki class $\mathcal{C}$). We hope to be able to address these matters in a future work.

It clearly suffices to prove Theorem 1.1 for a 1-dimensional base $\Delta \subset \mathbb{C}$ (i.e. an open disc in $\mathbb{C}$) that we can shrink at will about the origin. This choice of Δ will be implicit throughout the paper.

Regarding the method of this work, a word of explanation may be in order. On the face of it, it would seem that embedding all the projective fibres X_t with $t \neq 0$ into the same projective space (which is possible thanks, for example, to Siu's effective Matsusaka Big Theorem [Siu93]) might lead to a quick proof of Theorem 1.1. However, one would then run up against the difficulty of having to extend across the origin objects that are holomorphically defined on the punctured disc $\Delta^\star$. It is hard to see how this can be done without controlling the volumes of the projective submanifolds involved (which might *a priori* explode) near the origin. Such a uniform volume control would be equivalent to the uniform mass control obtained in Proposition 1.5, so one would be faced with the same difficulty as ours. Furthermore, the present method has the advantage of lending itself to generalisation when X_t is only assumed to be Kähler for $t \neq 0$ (cf. situation in Question 1.6).

Notation and terminology. A complex analytic family of compact complex manifolds will be often referred to simply as a *family* $(X_t)_{t\in\Delta}$. Given a smooth family of Hermitian metrics $(\gamma_t)_{t\in\Delta}$ on the fibres $(X_t)_{t\in\Delta}$, the formal adjoints $d_t^\star, \partial_t^\star, \bar\partial_t^\star$ associated with d, ∂_t and respectively $\bar\partial_t$ will be calculated with respect to the metric γ_t. They give rise to Laplace-Beltrami operators $\Delta_t = d\,d_t^\star + d_t^\star\,d$, $\Delta'_t = \partial_t\,\partial_t^\star + \partial_t^\star\,\partial_t$, $\Delta''_t = \bar\partial_t\,\bar\partial_t^\star + \bar\partial_t^\star\,\bar\partial_t$ acting on C^∞ forms of X of any degree $k = 1,\dots,n$ or any J_t-bidegree $(p,\,q)$, $p,q = 1,\dots,n$. The respective spaces of these forms will be denoted by $C^\infty_k(X,\,\mathbb{C})$ and $C^\infty_{p,q}(X_t,\,\mathbb{C})$. Given a form u, its component of type (p,q) with respect to the complex structure J_t will be denoted by $u_t^{p,q}$. The λ-eigenspace of $\Delta''_t : C^\infty_{p,q}(X_t,\,\mathbb{C}) \to C^\infty_{p,q}(X_t,\,\mathbb{C})$ will be denoted by $E^{p,q}_{\Delta''_t}(\lambda)$. Similarly for $\Delta'_t : C^\infty_{p,q}(X_t,\,\mathbb{C}) \to C^\infty_{p,q}(X_t,\,\mathbb{C})$. Dolbeault cohomology groups

of J_t-(p,q)-classes will be denoted by $H^{p,q}(X_t, \mathbb{C})$, while $H^k(X, \mathbb{C})$ will stand for De Rham cohomology. The respective dimensions of these $\mathbb{C}$-vector spaces are the usual Hodge numbers $h^{p,q}(t)$ and Betti numbers b_k. By the Kähler assumption on X_t for every $t \neq 0$, every $h^{p,q}(t)$ is constant on $\Delta^\star$ after possibly shrinking Δ about 0. But it may *a priori* happen that $h^{p,q}(0) > h^{p,q}(t)$ for $t \neq 0$, although this case is *a posteriori* ruled out by Theorem 1.1.

The $\partial\bar\partial$-lemma will be said to hold on a given compact complex manifold X if, for any C^∞ form u that is d-closed and of pure type (say (p, q)) on X, all the following exactness properties are equivalent for u :

$$u \text{ is } d\text{-exact} \iff u \text{ is } \partial\text{-exact} \iff u \text{ is } \bar\partial\text{-exact} \iff u \text{ is } \partial\bar\partial\text{-exact}.$$

It is well-known that the $\partial\bar\partial$-lemma holds on any compact Kähler manifold. We shall apply it in quite a number of instances on the fibres X_t with $t \neq 0$. One major difficulty in the proof of Theorem 1.1 stems from the $\partial\bar\partial$-lemma not being *a priori* known to hold on X_0, although this will be the case when Theorem 1.1 has been proved.

2 The special case of constant $h^{0,1}(t)$, $t \in \Delta$

In this section we prove the following special case of Theorem 1.1

Proposition 2.1 *Let $\pi : \mathcal{X} \to \Delta$ be a complex analytic family of compact complex manifolds such that the fibre X_t is projective for every $t \in \Delta^\star$. Suppose that $h^{0,1}(0) = h^{0,1}(t)$ for t close to 0. Then X_0 is Moishezon.*

The proof falls naturally into two steps.

Step 1 : produce a weak-limit current $T \geq 0$ on X_0 from $(\omega_t)_{t\in\Delta\setminus\Sigma}$

The proof of Gauduchon's theorem ([Gau77]) implies the existence of a smooth family of Gauduchon metrics on the fibres $X_t = (X, J_t)$. In other words, there exists a family of 2-forms $(\gamma_t)_{t\in\Delta}$ on X, varying in a C^∞ way with $t \in \Delta$, such that each γ_t is a positive-definite, type $(1, 1)$-form with respect to J_t and satisfies the Gauduchon condition on X_t : $\partial_t\bar\partial_t\gamma_t^{n-1} = 0$. To see this, let us briefly scan the argument of [Gau77] in our family context. Let $(\omega'_t)_{t\in\Delta}$ be any family of Hermitian metrics varying in a C^∞ way with t on $(X_t)_{t\in\Delta}$. Consider the Laplace-type operator acting on smooth functions :

$$P_{\omega'_t} := i\,\Lambda_{\omega'_t}\bar\partial_t\partial_t \,:\, C^\infty(X, \mathbb{C}) \to C^\infty(X, \mathbb{C}),$$

where $\Lambda_{\omega'_t}$ is the ω'_t-adjoint of the multiplication by ω'_t. The adjoint of $P_{\omega'_t}$ is

$$P^\star_{\omega'_t} \,:\, C^\infty(X, \mathbb{C}) \to C^\infty(X, \mathbb{C}), \quad P^\star_{\omega'_t}(f) = i \star \bar\partial_t\partial_t\bigg(f\,\frac{\omega_t'^{n-1}}{(n-1)!}\bigg),$$

where $\star = \star_{\omega'_t} : C^\infty_{n,n}(X_t, \mathbb{C}) \to C^\infty(X_t, \mathbb{C})$ is the Hodge-star operator (an isometry) associated with ω'_t. The operators $P_{\omega'_t}$ and $P^\star_{\omega'_t}$ are elliptic, ≥ 0, and of vanishing index (as the principal symbols are self-adjoint). Moreover, $\ker P_{\omega'_t} = \mathbb{C}$ (i.e. the constant functions) by the obvious inclusion $\mathbb{C} \subset \ker P_{\omega'_t}$ and the maximum principle. Hence, by ellipticity and vanishing index, $\dim \ker P^\star_{\omega'_t} = 1$. Furthermore, the proof of [Gau77] shows that any function $f \in \ker(P^\star_{\omega'_t|C^\infty(X,\mathbb{R})})$ must satisfy : $f > 0$ on X or $f < 0$ on X or $f = 0$ on X. The existence of a C^∞ function $f_t : X \to (0, +\infty)$ such that $P^\star_{\omega'_t}(f_t) = 0$ is equivalent to the Hermitian metric $f_t^{\frac{1}{n-1}}\,\omega'_t$ being Gauduchon. Now, $(P^\star_{\omega'_t})_{t\in\Delta}$ is a C^∞ family of elliptic operators on the fibres X_t with kernels of constant dimensions ($= 1$). By Kodaira and Spencer (see e.g. [Kod85, Theorem 7.4, p. 326]), the kernels define a C^∞ vector bundle $\Delta \ni t \mapsto \ker(P^\star_{\omega'_t})$. Then it suffices to pick $f_0 \in \ker(P^\star_{\omega'_0|C^\infty(X,\mathbb{R})})$ such that $f_0 > 0$ and to extend it to a C^∞ local section $\Delta \ni t \mapsto f_t$ of the C^∞ real bundle $\Delta \ni t \mapsto \ker(P^\star_{\omega'_t|C^\infty(X,\mathbb{R})})$ which is a trivial bundle if Δ has been shrunk sufficiently about 0. By continuity, $f_t > 0$ for all t sufficiently close to $0 \in \Delta$, defining a family $\gamma_t := f_t^{\frac{1}{n-1}}\,\omega'_t$, $t \in \Delta$, of Gauduchon metrics varying in a C^∞ way with t on the fibres X_t.

Fix any such family $(\gamma_t)_{t\in\Delta}$. It is against these Gauduchon metrics that the masses of forms will be measured. The following uniform mass boundedness proves Proposition 1.5 in the special case treated here.

Proposition 2.2 *Let $\alpha \in H^2(X, \mathbb{Z})$ a class given by Remark 1.2. For every $t \in \Delta \setminus \Sigma$, let ω_t be an arbitrary J_t-Kähler form belonging to the class α. If $h^{0,1}(t)$ is independent of $t \in \Delta$, there exists a constant $C > 0$ independent of $t \in \Delta \setminus \Sigma$ such that the masses of the ω_t's with respect to the γ_t^{n-1}'s satisfy :*

$$0 < \int\limits_{X_t} \omega_t \wedge \gamma_t^{n-1} \leq C < +\infty, \quad \textit{for all} \quad t \in \Delta \setminus \Sigma. \tag{7}$$

after possibly shrinking Δ about 0.

Notice that the choice (3) of Kähler forms ω_t in the given class α by means of the Aubin-Yau theorem is not needed here. It will come in later on.

Proof. The lower bound is obvious as $\omega_t > 0$ and $\gamma_t > 0$. Let $\tilde\omega$ be any d-closed real 2-form in the De Rham class α. As ω_t and $\tilde\omega$ are De Rham cohomologous real 2-forms, there exists a smooth real 1-form β_t on X_t such that :

$$\omega_t = \tilde\omega + d\beta_t \quad \text{on } X_t, \quad \text{for every} \quad t \in \Delta \setminus \Sigma. \tag{8}$$

Thus, for each $t \in \Delta \setminus \Sigma$, the mass of ω_t splits as :

$$\int\limits_{X_t} \omega_t \wedge \gamma_t^{n-1} = \int\limits_{X_t} \tilde\omega \wedge \gamma_t^{n-1} + \int\limits_{X_t} d\beta_t \wedge \gamma_t^{n-1}. \tag{9}$$

As the forms $(\gamma_t)_{t\in\Delta}$ vary in a C^∞ way with $t\in\Delta$, the first term in the right-hand side of (9) is bounded as t varies in a neighbourhood of 0. We are thus reduced to showing the boundedness of the second term as $t\in\Delta\setminus\Sigma$ approaches 0. The difficulty stems from the fact that the family $(\omega_t)_{t\in\Delta\setminus\Sigma}$ of Kähler forms (and implicitly $(\beta_t)_{t\in\Delta\setminus\Sigma}$) need not extend to the limit fibre X_0 as X_0 is not assumed to be Kähler. By Stokes' theorem we get :

$$\int\limits_{X_t} d\beta_t\wedge\gamma_t^{n-1} = \int\limits_{X_t}(\partial_t\beta_t^{0,1}+\bar\partial_t\beta_t^{1,0})\wedge\gamma_t^{n-1} = -\int\limits_{X_t}\beta_t^{1,0}\wedge\bar\partial_t\gamma_t^{n-1} - \int\limits_{X_t}\beta_t^{0,1}\wedge\partial_t\gamma_t^{n-1}, \tag{10}$$

where $\beta_t=\beta_t^{1,0}+\beta_t^{0,1}$ is the decomposition of β_t into components of types $(1,0)$ and $(0,1)$. As $\beta_t=\overline{\beta_t}$ (i.e. β_t is a real form), $\beta_t^{1,0}=\overline{\beta_t^{0,1}}$ and the two terms in the right-hand side above are conjugate to each other. It thus suffices to show the boundedness of the integral containing $\beta_t^{0,1}$ as $t\in\Delta\setminus\Sigma$ approaches 0.

Now the solution β_t of equation (8) is not unique. We will make a particular choice of β_t. The Kähler form ω_t being of J_t-type $(1,1)$, equating the components of J_t-type $(0,2)$ in (8), we see that $\beta_t^{0,1}$ must solve the equation :

$$\bar\partial_t\beta_t^{0,1} = -\tilde\omega_t^{\,0,2} \quad \text{on} \quad X_t, \quad \text{for} \quad t\in\Delta\setminus\Sigma. \tag{11}$$

Conversely, for every $t\in\Delta^\star$, choose $\beta_t^{0,1}$ to be the solution of equation (11) of minimal L^2 norm with respect to the metric γ_t of X_t. (Notice that equation (11) is solvable in $\beta_t^{0,1}$ for every $t\neq 0$ because α contains J_t-type (1, 1)-form for every $t\neq 0$. However, it need not be solvable for $t=0$ as α is not known to contain a J_0-type (1, 1)-form.) Set $\beta_t^{1,0}:=\overline{\beta_t^{0,1}}$ and $\beta_t:=\beta_t^{1,0}+\beta_t^{0,1}$. Clearly, β_t is a real 1-form on X but it need not solve equation (8). However, the $\partial\bar\partial$-Lemma (which holds on every X_t with $t\neq 0$ by the Kähler assumption) shows that β_t satisfies equation (8) on X_t, for each $t\in\Delta\setminus\Sigma$, up to a $\partial_t\bar\partial_t$-exact (1, 1)-form. Indeed, $\tilde\omega+d\,\beta_t$ is J_t-type (1, 1) since its (0, 2)-component is $\tilde\omega_t^{0,2}+\bar\partial_t\beta_t^{0,1}=0$ by (11) and its (2, 0)-component also vanishes by conjugation. It follows that $\tilde\omega_t+d\,\beta_t-\omega_t$ is of J_t-type (1, 1) and d-exact, hence also $\partial_t\bar\partial_t$-exact by the $\partial\bar\partial$-Lemma. Thus β_t solves the equation :

$$\omega_t=\tilde\omega_t+d\beta_t+i\partial_t\bar\partial_t\varphi_t \quad \text{on } X_t, \quad \text{for} \quad t\in\Delta\setminus\Sigma, \tag{12}$$

for some smooth function φ_t on X. Now, since γ_t has been chosen such that $\partial_t\bar\partial_t\gamma_t^{n-1}=0$ (the Gauduchon condition), Stokes' theorem gives :

$$\int\limits_{X_t} i\partial_t\bar\partial_t\varphi_t\wedge\gamma_t^{n-1} = \int\limits_{X_t}\varphi_t\wedge i\partial_t\bar\partial_t\gamma_t^{n-1}=0.$$

In other words, $\partial\bar{\partial}$-exact (1, 1)-forms have no mass against the relevant power of a Gauduchon form. Thus relation (9) holds thanks to (12) and, as explained above, the proof reduces to showing the boundedness of the integral containing $\beta_t^{0,\,1}$ in the right-hand side of (10) as $t \in \Delta^\star$ approaches 0.

For every $t \in \Delta$, let $\bar{\partial}_t^\star$ denote the formal adjoint of $\bar{\partial}_t$ with respect to the global L^2 scalar product defined by the Gauduchon metric γ_t of X_t. We get a C^∞ family $(\Delta''_t)_{t\in\Delta}$ of associated anti-holomorphic Laplace-Beltrami operators defined as

$$\Delta''_t = \bar{\partial}_t\,\bar{\partial}_t^\star + \bar{\partial}_t^\star\,\bar{\partial}_t \quad \text{on} \quad X_t = (X,\,J_t), \quad t \in \Delta.$$

The (unique) minimal L^2 solution of equation (11) is known to be given by the formula :

$$\beta_t^{0,1} = -G_t\,\bar{\partial}_t^\star\tilde{\omega}_t^{\,0,2}, \qquad t \in \Delta\setminus\Sigma, \tag{13}$$

where G_t denotes the Green operator of Δ''_t. Clearly, the family of operators $(\bar{\partial}_t^\star)_{t\in\Delta}$ varies in a C^∞ way with t. By the Hodge Fundamental Theorem (which does not require the Kähler property), the Hodge isomorphism holds :

$$H^{0,1}(X_t,\mathbb{C}) \simeq \mathcal{H}^{0,1}(X_t,\mathbb{C}), \qquad t \in \Delta,$$

where $\mathcal{H}^{0,1}(X_t,\mathbb{C}) := \ker\Delta''_t$ is the space of harmonic J_t-$(0,1)$-forms. By a well-known result of Kodaira and Spencer (see [Kod86, Theorem 7.6, p. 344]), the family of Green operators $(G_t)_{t\in\Delta}$ of a C^∞ family of strongly elliptic operators $(\Delta''_t)_{t\in\Delta}$ is C^∞ with respect to $t \in \Delta$ if the dimensions of the kernel spaces $\mathcal{H}^{0,1}(X_t,\mathbb{C})$ are independent of $t \in \Delta$. This is indeed the case here as, by assumption, $h^{0,1}(t) = \dim H^{0,1}(X_t,\,\mathbb{C})$ is independent of $t \in \Delta$, and $\dim\mathcal{H}^{0,1}(X_t,\mathbb{C}) = h^{0,1}(t)$ by the Hodge isomorphism.

Now the J_t-type (0, 2)-components $(\tilde{\omega}_t^{\,0,2})_{t\in\Delta}$ of the fixed 2-form $\tilde{\omega}$ vary in a C^∞ way with t (up to $t = 0$) since the complex structures $(J_t)_{t\in\Delta}$ do. As the composed operators $G_t\,\bar{\partial}_t^\star$ have the same property, the forms $\beta_t^{0,1} = -G_t\,\bar{\partial}_t^\star\tilde{\omega}_t^{\,0,2}$ (cf. (13)) extend smoothly across $t = 0$ to a family $(\beta_t^{0,1})_{t\in\Delta}$ of forms which vary in a C^∞ way with $t \in \Delta$. This clearly implies the boundedness in a neighbourhood of $t = 0$ of the second term in the right-hand side of (10). Taking conjugates, the same is true of the first term in the right-hand side of (10). This completes the proof of Proposition 2.2. $\square$

As the family of positive forms $(\omega_t)_{t\in\Delta\setminus\Sigma}$ is bounded in mass, it is weakly compact. Thus it contains a weakly convergent subsequence $\omega_{t_k} \to T$, with $\Delta\setminus\Sigma \ni t_k \to 0$ as $k \to +\infty$. The limit current $T \geq 0$ is closed, positive and of type $(1,1)$ for the limit complex structure J_0 of X_0. By the weak continuity of De Rham classes $\{\ \}$, $\{\omega_{t_k}\} \to \{T\}$. As $\{\omega_{t_k}\} = \alpha$ for all k, we see that $T \in \alpha$.

We have thus produced a closed positive $(1,1)$-current $T \geq 0$ on X_0 in the given integral De Rham cohomology class α.

Step 2 : prove that the integral class $\{T\}$ contains a Kähler current

This is where the singular Morse inequalities for integral classes (Theorem 1.3) come into play. The semicontinuity property of the absolutely continuous part of currents (see e.g. [Bou02]) spells :

$$T_{ac}(x)^n \geq \limsup_{k\to+\infty} \omega_{t_k}(x)^n, \quad \text{for almost every} \quad x \in X_0. \tag{14}$$

Now, if the Kähler forms ω_t, $t \in \Delta \setminus \Sigma$, are chosen in the given Kähler class α by means of the Aubin-Yau theorem ([Yau78]) as explained in (3), the identity $\omega_{t_k}^n = v\, dV_{t_k}$ and (14) give :

$$T_{ac}(x)^n \geq v \limsup_{k\to+\infty} dV_{t_k}(x) = v\, dV_0(x), \quad \text{for almost every } x \in X_0. \text{ Hence}$$

$$\int\limits_{X_0} T_{ac}^n \geq v > 0. \tag{15}$$

In particular, T satisfies condition (ii) of Theorem 1.3. It is for this sole purpose that the Aubin-Yau [Yau78] theorem has been used.

Summing up, the limit current T is of type (1, 1) (for J_0), has an integral De Rham cohomology class α and satisfies the mild positivity assumptions (i) and (ii) of Theorem 1.3 on singular Morse inequalities. By that theorem applied on X_0, α must contain a Kähler current, hence X_0 must be Moishezon.

The proof of Proposition 2.1 is complete. $\square$

Remark 2.3 When trying to dispense with the non-jumping hypothesis that was made in Propositions 2.1 and 2.2 on $h^{0,1}(t)$ at $t=0$, one is faced with the following difficulty in proving the existence of a uniform upper bound (25) for the masses of the Kähler forms $(\omega_t)_{t\in\Delta\setminus\Sigma}$. For every $t \in \Delta$, the Laplace operator Δ''_t acting on J_t-(0, 1)-forms of X is elliptic and therefore has a compact resolvent and a discrete spectrum

$$0 = \lambda_0(t) \leq \lambda_1(t) \leq \cdots \leq \lambda_k(t) \leq \dots \tag{16}$$

with $\lambda_k(t) \to +\infty$ as $k \to +\infty$. By the Hodge isomorphism, the multiplicity of zero as an eigenvalue of Δ''_t equals $h^{0,1}(t)$. By results of Kodaira and Spencer (see [Kod85, Lemmas 7.5-7.7 and Proof of Theorem 7.2, p. 338-343]), for every small $\varepsilon > 0$, the number $m \in \mathbb{N}^\star$ of eigenvalues (counted with multiplicities) of Δ''_t contained in the interval $[0, \varepsilon)$ is independent of t if $t \in \Delta$ is sufficiently close to 0 (say δ_ε-close). If $\varepsilon > 0$ has been chosen so small that 0 is the only eigenvalue of Δ''_0 contained in $[0, \varepsilon)$, it follows that $m = h^{0,1}(0) \geq h^{0,1}(t)$ for t sufficiently close to 0 (the upper-semicontinuity

property). Consequently, for t near 0, $h^{0,1}(0) = h^{0,1}(t)$ if and only if 0 is the only eigenvalue of Δ''_t lying in $[0, \varepsilon)$. In other words, if $h^{0,1}(0) > h^{0,1}(t)$ when $t(\neq 0)$ is near 0, choosing increasingly small $\varepsilon > 0$ gives eigenvalues of Δ''_t :

$$0 < \lambda_{k_1}(t) \leq \lambda_{k_2(t)} \leq \cdots \leq \lambda_{k_N}(t) := \varepsilon_t < \varepsilon, \quad t \in \Delta^\star, \tag{17}$$

that converge to zero (i.e. $\varepsilon_t \to 0$) when $t \to 0$, where $N = h^{0,1}(0) - h^{0,1}(t)$. Now, formula (13) for $\beta_t^{0,1}$ involves the Green operator G_t which is the inverse of the restriction of Δ''_t to the orthogonal complement of its kernel. The inverses $1/\lambda_{k_j}(t) \to +\infty$ of the small eigenvalues of Δ''_t are eigenvalues for G_t. Thus, if $\bar\partial_t^\star \tilde\omega_t^{0,2}$ has non-trivial projections onto the eigenspaces $E^{0,1}_{\Delta''_t}(\lambda_{k_j(t)})$, these projections get multiplied by $1/\lambda_{k_j}(t)$ when G_t acts on $\bar\partial_t^\star \tilde\omega_t^{0,2}$. Then $\beta_t^{0,1}$ need not be bounded as t approaches 0, unless the said projections can be proved to tend to zero sufficiently quickly to offset $1/\lambda_{k_j}(t) \to +\infty$ when t approaches 0. This may cause the mass of ω_t (cf. (9)) to get arbitrarily large in the limit as $t \to 0$.

Thus the remaining difficulty in proving Theorem 1.1 is to prove the uniform mass boundedness of Proposition 2.2 without the non-jumping assumption on $h^{0,1}(t)$. The rest of the paper will be devoted to solving this difficulty.

3 The strongly Gauduchon special case

In this section we shall exhibit a different kind of hypothesis under which Theorem 1.1 can be proved comparatively painlessly. We have deemed it necessary to include this discussion as the method introduced here will be developed in the next section to give the general case of Theorem 1.1.

Setting the method in motion

The notation is carried forward from the previous sections. Fix any family $(\gamma_t)_{t\in\Delta}$ of J_t-Gauduchon metrics varying in a C^∞ way with t. As explained after the identity (10), *Step 1* (hence everything that follows) in the proof of Theorem 1.1 can be run if we can guarantee the boundedness of the following integral, that will henceforth be termed the *main quantity* :

$$I_t := \int\limits_{X_t} \partial_t \beta_t^{0,1} \wedge \gamma_t^{n-1} = -\int\limits_{X_t} \beta_t^{0,1} \wedge \partial_t \gamma_t^{n-1}, \quad t \in \Delta^\star, \tag{18}$$

as t approaches 0. The difficulty is that the family $(\beta_t^{0,1})_{t\in\Delta^\star}$ of $J_t-(0,1)$-forms constructed as the minimal L^2 solutions of equations (11) (extended to all $t \neq 0$) need not be bounded as t approaches 0 if $h^{0,1}(0) > h^{0,1}(t), t \neq 0$ (cf. Remark 2.3). Thus, $\partial_t \beta_t^{0,1}$ may "explode" near $t = 0$. However, $\bar\partial_t \beta_t^{0,1} = \tilde\omega_t^{0,2}$ is bounded and extends smoothly to $t = 0$ by construction, since the family

$(\tilde{\omega}_t^{0,2})_{t\in\Delta}$ of $J_t-(0,2)$-components of the fixed 2-form $\tilde{\omega}$ varies in a C^∞ way with $t\in\Delta$. This means that if we can convert $\partial_t\beta_t^{0,1}$ to $\bar\partial_t\beta_t^{0,1}$ in (18), we will obtain the desired boundedness of the *main quantity* I_t, hence Proposition 1.5 and implicitly a proof of Theorem 1.1. This very simple observation is the starting point of our method to tackle the general case.

The next observation is that the $(n,n-1)$-form $\partial_t\gamma_t^{n-1}$ is d-closed. Indeed, it is ∂_t-closed in a trivial way and is $\bar\partial_t$-closed by the Gauduchon assumption on γ_t. For $t\neq 0$, the $\partial\bar\partial$-lemma holds on X_t (thanks to the Kähler assumption) and yields the $\bar\partial_t$-exactness of $\partial_t\gamma_t^{n-1}$. Thus,

$$\partial_t\gamma_t^{n-1}=\bar\partial_t\zeta_t, \qquad t\neq 0, \tag{19}$$

where the C^∞ J_t-$(n,n-2)$-form ζ_t can be chosen as the minimal L^2 solution of the above equation (with respect to γ_t), generating a C^∞ family $(\zeta_t)_{t\in\Delta^\star}$ defined off $t=0$. By Stokes' theorem, the *main quantity* now reads :

$$I_t=-\int\limits_{X_t}\beta_t^{0,1}\wedge\partial_t\gamma_t^{n-1}=\int\limits_{X_t}\bar\partial_t\beta_t^{0,1}\wedge\zeta_t, \qquad t\in\Delta^\star. \tag{20}$$

The situation is now the reverse of that in (18) : we have rendered the factor depending on $\beta_t^{0,1}$ bounded, as $\bar\partial_t\beta_t^{0,1}=\tilde{\omega}_t^{0,2}$ varies in a C^∞ way with t up to $t=0$, but we are now faced with the task of ensuring the boundedness of the other factor ζ_t near $t=0$. The difficulty stems from the fact that, at this point, the $\partial\bar\partial$-lemma is not known to hold on X_0. However, if $\partial_0\gamma_0^{n-1}$ were known to be $\bar\partial_0$-exact, the proof of Theorem 1.1 could be completed.

We will now highlight a kind of hypothesis on X_0, different to the one considered in the previous section, that guarantees the $\bar\partial_0$-exactness of $\partial_0\gamma_0^{n-1}$. We digress briefly to introduce a new type of Gauduchon metrics satisfying an extra property.

Strongly Gauduchon metrics

Definition 3.1 *Let X be a compact complex manifold, $\dim_{\mathbb{C}}X=n$.*

(i) A C^∞ positive-definite $(1,1)$-form γ on X will be said to be a **strongly Gauduchon metric** *if the $(n,n-1)$-form $\partial\gamma^{n-1}$ is $\bar\partial$-exact on X.*

(ii) If X carries such a metric, X will be said to be a **strongly Gauduchon manifold**.

Notice that the Gauduchon condition only requires $\partial\gamma^{n-1}$ to be $\bar\partial$-closed on X. Hence, every *strongly Gauduchon* metric is a Gauduchon metric. Now, if the $\partial\bar\partial$-lemma holds on X (as is the case if, for example, X is Kähler), the converse statement holds as well (see argument above), and therefore the two notions coincide in that case. However, we will now show that the *strongly Gauduchon* condition is strictly stronger than the Gauduchon condition in general. Furthermore, unlike Gauduchon metrics which exist on any compact complex manifold, *strongly Gauduchon* metrics need not exist in general. We

will give a necessary and sufficient condition on the manifold X ensuring the existence of a *strongly Gauduchon* metric. The method, proceeding by duality and an application of the Hahn-Banach separation theorem in locally convex spaces, is the classical one introduced by Sullivan in [Sul76] and used in several instances in [HL83], [Mic83], [Lam99], [Buc99].

We begin with the following very simple observation.

Lemma 3.2 *A complex manifold X of complex dimension n carries a strongly Gauduchon metric γ if and only if there exists a real d-closed C^∞ form Ω of degree $2n-2$ on X such that its component of type $(n-1,n-1)$ satisfies $\Omega^{n-1,n-1}>0$ on X.*

Proof. For a real $(2n-2)$-form Ω, the $(2n-1)$-form $d\Omega$ is also real, hence its components of type $(n,n-1)$ and respectively $(n-1,n)$ are conjugate to each other. Thus, the condition $d\Omega=0$ amounts to

$$\partial\Omega^{n-1,n-1}=-\bar\partial\Omega^{n,n-2}. \tag{21}$$

If a *strongly Gauduchon* metric γ exists on X, we set $\Omega^{n-1,n-1}:=\gamma^{n-1}$. This is a smooth form of type $(n-1,n-1)$ and $\Omega^{n-1,n-1}>0$. By the *strongly Gauduchon* condition on γ, $\partial\Omega^{n-1,n-1}$ is $\bar\partial$-exact on X. Hence, one can find a smooth form $\Omega^{n,n-2}$ of type $(n,n-2)$ on X such that $\partial\Omega^{n-1,n-1}=-\bar\partial\Omega^{n,n-2}$. By setting $\Omega^{n-2,n}:=\overline{\Omega^{n,n-2}}$ and $\Omega=\Omega^{n,n-2}+\Omega^{n-1,n-1}+\Omega^{n-2,n}$, we get the desired form of degree $2n-2$.

Conversely, if there exists a $(2n-2)$-form Ω on X as in the statement, the assumption $\Omega^{n-1,n-1}>0$ allows one to extract the root of order $n-1$ in the following sense. A very useful remark of Michelsohn [Mic83, p.279-280] in linear algebra asserts that there is a unique positive-definite smooth form γ of type $(1,1)$ on X such that $\Omega^{n-1,n-1}=\gamma^{n-1}$. By the assumption $d\Omega=0$ and its equivalent formulation (21), we see that γ satisfies the *strongly Gauduchon* condition. $\square$

We shall now determine when a $(2n-2)$-form as in Lemma 3.2 above exists. Let X be any compact complex manifold, $\dim_{\mathbb{C}}X=n$, and let Ω be any C^∞ form of degree $2n-2$ on X. The condition $d\Omega=0$ is equivalent, by the duality between d-closed smooth real $(2n-2)$-forms and real exact 2-currents $T=d\,S$ on X, to the property

$$\int\limits_X\Omega\wedge d\,S=0,\quad\text{for every real 1–current } S \text{ on } X. \tag{22}$$

On the other hand, the duality between strictly positive, smooth $(n-1,n-1)$-forms and non-zero positive $(1,1)$-currents on X shows that the condition $\Omega^{n-1,n-1}>0$ is equivalent to the property

$$\int_X \Omega^{n-1,n-1} \wedge T > 0, \quad \text{for every non-zero } (1,1)\text{–current } T \geq 0 \text{ on } X. \quad (23)$$

Now, if T is of type $(1,1)$, we clearly have $\int_X \Omega^{n-1,n-1} \wedge T = \int_X \Omega \wedge T$. Furthermore, real d-exact 2-currents $T = d\,S$ form a closed vector subspace $\mathcal{A}$ of the locally convex space $\mathcal{D}'_{\mathbb{R}}(X)$ of real 2-currents on X. Meanwhile, if we fix a smooth, strictly positive $(n-1,n-1)$-form Θ on X, positive non-zero $(1,1)$-currents T on X can be normalised such that $\int_X T \wedge \Theta = 1$ and it suffices to guarantee property (23) for normalised currents. Clearly, these normalised positive $(1,1)$-currents form a compact (in the locally convex topology of weak convergence of currents) convex subset $\mathcal{B}$ of the locally convex space $\mathcal{D}'_{\mathbb{R}}(X)$ of real 2-currents on X. The Hahn-Banach separation theorem for locally convex spaces (see [HL83] and the references given there) guarantees the existence of a linear functional vanishing identically on a given closed subset and assuming only positive values on a given compact subset if the two subsets are convex and do not intersect. Hence, in our case, there exists a real smooth $(2n-2)$-form Ω on X satisfying both conditions (22) and (23) if and only if $\mathcal{A} \cap \mathcal{B} = \emptyset$. This amounts to there existing no non-trivial exact $(1,1)$-current $T = dS$ such that $T \geq 0$ on X. We have thus proved (cf. Lemma 3.2) the following characterisation of *strongly Gauduchon* manifolds in terms of non-existence of certain currents. This closely parallels similar existence criteria for Kähler metrics ([HL83]) and Michelsohn's balanced metrics ([Mic83]).

Proposition 3.3 *Let X be a compact complex manifold, $\dim_{\mathbb{C}} X = n$. Then, X carries a strongly Gauduchon metric γ if and only if there is no non-zero current T of type $(1,1)$ on X such that $T \geq 0$ and T is d-exact on X.*

We now end this digression on a few simple remarks. Given a Hermitian metric (equivalently, a C^∞ positive-definite $(1,1)$-form) γ on a compact complex manifold X, the following four conditions on γ : the Kähler condition ($d\gamma = 0$), Michelsohn's balanced condition ($d^\star\gamma = 0$) , the *strongly Gauduchon* condition ($\partial\gamma^{n-1}$ is $\bar\partial$-exact) and the Gauduchon condition ($\partial\gamma^{n-1}$ is $\bar\partial$-closed) stand in the following implication hierarchy :

$$\text{Kähler} \Longrightarrow \text{balanced} \Longrightarrow \text{strongly Gauduchon} \Longrightarrow \text{Gauduchon} \qquad (24)$$

For example, the implication "balanced $\Longrightarrow$ strongly Gauduchon" can be seen as follows. Using the Hodge $\star$ operator that gives isometries $\star : \Lambda^{p,q}T^\star X \longrightarrow \Lambda^{n-q,n-p}T^\star X$ defined by the Hermitian metric γ on X, we have $\star\gamma = \gamma^{n-1}/(n-1)!$ and $d^\star = -\star\, d\, \star$. Hence, the balanced condition $d^\star\gamma = 0$

is equivalent to $d\gamma^{n-1}=0$, which in turn is equivalent, thanks to γ^{n-1} being a real form, to $\partial\gamma^{n-1}=0$ (cf. [Mic83]). This is clearly a stronger condition than the *strongly Gauduchon* requirement that $\partial\gamma^{n-1}$ be $\bar\partial$-exact.

Except for Gauduchon metrics, any of the other three kinds of metrics in (24) need not exist in general. Each of the conditions (24) can be given an intrinsic characterisation in terms of non-existence of certain currents (cf. [HL83], [Mic83], Proposition 3.3 above, for the first three of them respectively). Recall that the fourth condition is known to have a similar characterisation that can be obtained by the same method : a Gauduchon metric γ exists on X if and only if there is no non-zero $(1,1)$-current T that is both positive and $\partial\bar\partial$-exact, i.e. $T=i\partial\bar\partial\varphi\geq 0$ globally on X. The compactness assumption on X and the maximum principle for psh functions rule out the existence of such a current, proving that a Gauduchon metric always exists on any compact complex manifold.

On manifolds of complex dimension ≥ 3, the implications (24) are strict. However, on compact complex surfaces the notions of Kähler and balanced metrics are equivalent ([Mic83]) and so are the notions of Kähler, balanced and *strongly Gauduchon* surfaces (i.e. surfaces carrying the respective kind of metrics). Indeed, it is well-known that a compact complex surface is Kähler if and only if its first Betti number b_1 is even (see [Siu83] and also [Buc99], [Lam99]). Now, it can be easily shown by the same duality method of Sullivan (see, e.g. [Lam99, Théorème 6.1]) that a current as described in Proposition 3.3 always exists on any compact complex surface with b_1 odd.

Proof of Theorem 1.1 under the **strongly Gauduchon** *assumption on* X_0

We now pick up where we left off before Definition 3.1. As hinted there, the proof of Theorem 1.1 would be complete if we were able to choose our family of Gauduchon metrics $(\gamma_t)_{t\in\Delta}$, varying in a C^∞ way with t, such that γ_0 is a *strongly Gauduchon* metric on X_0. Indeed, the above preparations being understood, Proposition 1.5 can be proved under the present circumstances.

Proposition 3.4 *Suppose the limit fibre X_0 of a family as in Theorem 1.1 is a strongly Gauduchon manifold. Then, after possibly shrinking Δ about 0, there exists a family $(\gamma_t)_{t\in\Delta}$, varying in a C^∞ way with t, of* **strongly Gauduchon** *metrics on the fibres $(X_t)_{t\in\Delta}$. Implicitly, uniform mass boundedness holds :*

$$0<\int\limits_{X_t}\omega_t\wedge\gamma_t^{n-1}\leq C<+\infty, \qquad \textit{for all} \quad t\in\Delta\setminus\Sigma, \tag{25}$$

where, for every $t\in\Delta\setminus\Sigma$, ω_t is any J_t-Kähler form belonging to the class $\alpha\in H^2(X,\mathbb{Z})$ given by Remark 1.2 and $C>0$ is a constant independent of $t\in\Delta\setminus\Sigma$.

Proof. As the limit fibre X_0 is assumed to be *strongly Gauduchon*, by the above Proposition 3.3 there is no current as described there on X_0. Equivalently, there exists a real smooth $(2n-2)$-form Ω on X such that Ω is d-closed and $\Omega_0^{n-1,\,n-1} > 0$ on X_0 (cf. Lemma 3.2). Now, the J_t-type-$(n-1, n-1)$-components of Ω vary in a C^∞ way with $t \in \Delta$ as the complex structures $(J_t)_{t\in\Delta}$ do. Thus, after possibly shrinking Δ about 0, we still have $\Omega_t^{n-1,\,n-1} > 0$ on X_t. Furthermore, Michelsohn's procedure for extracting the root of order $n-1$ being a purely linear-algebraic argument, the family of corresponding roots $(\gamma_t)_{t\in\Delta}$ (i.e. $\gamma_t^{n-1} = \Omega_t^{n-1,\,n-1}$) varies in a C^∞ way with t. It is therefore a C^∞ family of *strongly Gauduchon* metrics. Moreover, as $d\,\Omega = 0$, (21) reads :

$$\partial_t \gamma_t^{n-1} = -\bar\partial_t \Omega_t^{n,\,n-2}, \qquad t \in \Delta. \tag{26}$$

Thus (20) shows that the *main quantity* I_t extends to $t = 0$ and reads

$$I_t = -\int\limits_{X_t} \bar\partial_t \beta_t^{0,\,1} \wedge \Omega_t^{n,\,n-2} = -\int\limits_{X_t} \tilde\omega_t^{0,\,2} \wedge \Omega_t^{n,\,n-2}, \qquad t \in \Delta. \tag{27}$$

As the family $(\Omega_t^{n,\,n-2})_{t\in\Delta}$ of $J_t - (n,\,n-2)$-components of the fixed form Ω varies in a C^∞ way with t, so does the family $(I_t)_{t\in\Delta}$. In view of the explanations given at the beginning of this section, the proof is complete. $\Box$

The above arguments add up to the following special case of Theorem 1.1 that we have been aiming at throughout this section.

Proposition 3.5 *Let $\mathcal{X} \to \Delta$ be a complex analytic family of compact complex manifolds such that the fibre $X_t := \pi^{-1}(t)$ is projective for every $t \in \Delta^\star$. Suppose that there does not exist any non-zero current of J_0-type $(1,1)$ which is both d-exact and ≥ 0 on X_0 (equivalently, X_0 is a strongly Gauduchon manifold). Then X_0 is Moishezon.*

The *strongly Gauduchon* assumption on X_0 is different in nature to the non-jumping assumption made on $h^{0,1}(t)$ in the previous section. As the currents whose existence is ruled out by the *strongly Gauduchon* assumption are rather exceptional, this does not appear to be too strong a hypothesis when the complex dimension of the fibres is ≥ 3. In the case of families of complex surfaces, the *strongly Gauduchon* assumption on X_0 amounts to the Kähler assumption which, clearly, we have no interest in making. However, as noticed in the Introduction, the surface-fibre case of Theorem 1.1 follows from well-known facts in the theory of compact complex surfaces and the arguments given in Section 2. The limit surface X_0 is even projective.

4 The general case

To prove the general case of Theorem 1.1, we now show that the situation considered in Section 3 always occurs under the mere assumption that the $\partial\bar\partial$-lemma hold (see terminology spelt out in the Introduction) on every fibre X_t with $t \neq 0$. This hypothesis is weaker than the Kähler, and so much more so than the projective, assumption.

Proposition 4.1 *Let $\pi : \mathcal{X} \rightarrow \Delta$ be a complex analytic family of compact complex manifolds such that the $\partial\bar\partial$-lemma holds on the fibre $X_t := \pi^{-1}(t)$ for every $t \in \Delta^\star := \Delta \setminus \{0\}$. Then X_0 is a* **strongly Gauduchon manifold**.

It is clear that the combined Propositions 3.5 and 4.1 prove Theorem 1.1. The object of this section is to give a proof to Proposition 4.1 whose hypothesis is henceforth supposed to hold. To this end, we will show that any family $(\gamma_t)_{t\in\Delta}$ of Gauduchon metrics varying in a C^∞ way with t can be modified to a family $(\rho_t)_{t\in\Delta}$ of *strongly Gauduchon* metrics varying in a C^∞ way with t.

Reduction of the uniform boundedness problem to a positivity problem

Fix any C^∞ family $(\gamma_t)_{t\in\Delta}$ of Gauduchon metrics on the respective fibres $(X_t)_{t\in\Delta}$. Denote Δ_t, Δ'_t and Δ''_t the Laplace-Beltrami operators (see Introduction) induced by the metrics γ_t on X_t. Let, as in (16), $(\lambda_j(t))_{j\in\mathbb{N}}$ denote the eigenvalues, ordered non-increasingly and repeated as many times as the respective multiplicity, of

$$\Delta''_t : C^\infty_{n,n-1}(X_t, \mathbb{C}) \longrightarrow C^\infty_{n,n-1}(X_t, \mathbb{C}), \qquad t \in \Delta.$$

By [Kod86], each λ_j is a continuous function of $t \in \Delta$. If there are eigenvalues such that $\lambda_j(t) > 0$ for $t \neq 0$ and $\lambda_j(0) = 0$, there are only finitely many of them numbering $h^{n,\,n-1}(0) - h^{n,\,n-1}(t) = h^{0,1}(0) - h^{0,1}(t)$ for any $t \neq 0$ close to 0. This number is, of course, independent of $t \neq 0$. For $t \neq 0$, let $\varepsilon_t > 0$ denote the largest of these *small* eigenvalues, so $\varepsilon_t \rightarrow 0$ as $t \rightarrow 0$. The remaining, infinitely many, eigenvalues are then bounded below (after possibly shrinking Δ about 0) by some $\varepsilon > 0$ independent of $t \in \Delta$. Thus,

$$\operatorname{Spec} \Delta''_t \subset [0,\ \varepsilon_t] \cup [\varepsilon,\ +\infty), \qquad t \in \Delta, \tag{28}$$

and we get an orthogonal eigenspace decomposition :

$$C^\infty_{n,n-1}(X_t, \mathbb{C}) = \bigoplus_{\lambda \leq \varepsilon_t} E^{n,n-1}_{\Delta''_t}(\lambda) \oplus \bigoplus_{\lambda \geq \varepsilon} E^{n,n-1}_{\Delta''_t}(\lambda), \qquad t \in \Delta. \tag{29}$$

Now, Δ''_t being an elliptic self-adjoint operator, it has a compact resolvent and there exists an orthonormal basis $(e^{n,n-1}_j(t))_{j\in\mathbb{N}}$ of $C^\infty_{n,n-1}(X_t, \mathbb{C})$ consisting of eigenvectors of Δ''_t :

$$\Delta''_t\, e_j^{n,n-1}(t) = \lambda_j(t)\, e_j^{n,n-1}(t), \qquad t\in\Delta. \tag{30}$$

Furthermore, in the three-space orthogonal decomposition

$$C^\infty_{n,n-1}(X_t,\mathbb{C}) = \ker\Delta''_t \oplus \operatorname{Im}\bar\partial_t \oplus \operatorname{Im}\bar\partial_t^\star, \tag{31}$$

each subspace is Δ''_t-invariant due to Δ''_t commuting with $\bar\partial_t$ and $\bar\partial_t^\star$. This means that the eigenvectors $e_j^{n,n-1}(t)$ forming an orthonormal basis can be chosen such that each of them lies in one (and only one) of the three subspaces of (31). So none of the $e_j^{n,n-1}(t)$ straddles two or three subspaces. These simple reductions are valid for every $t\in\Delta$ and we will henceforth suppose that the choices have been made as described above. The orthogonal decomposition of $\partial_t\gamma_t^{n-1}\in C^\infty_{n,n-1}(X_t,\mathbb{C})$ according to (29) has the shape :

$$\partial_t\gamma_t^{n-1} = \sum_{j\in J_1} c_j(t)\, e_j^{n,n-1}(t) + \sum_{j\in J_2} c_j(t)\, e_j^{n,n-1}(t) = U_t + V_t, \qquad t\in\Delta, \tag{32}$$

where $U_t = \sum\limits_{j\in J_1} c_j(t)\, e_j^{n,n-1}(t) \in \bigoplus\limits_{\lambda\leq\varepsilon_t} E^{n,n-1}_{\Delta''_t}(\lambda)$ and $V_t = \sum\limits_{j\in J_2} c_j(t)\, e_j^{n,n-1}(t) \in \bigoplus\limits_{\lambda\geq\varepsilon} E^{n,n-1}_{\Delta''_t}(\lambda)$, with coefficients $c_j(t)\in\mathbb{C}^\star$ and index sets $J_1, J_2\subset\mathbb{N}$ such that $J_1\cap J_2=\emptyset$. As already noticed, by the Gauduchon condition, $\partial_t\gamma_t^{n-1}$ is d-closed for all $t\in\Delta$ and, since it is ∂_t-exact, it must also be $\bar\partial_t$-exact for all $t\neq 0$ by the $\partial\bar\partial$-lemma. Since each eigenvector $e_j^{n,n-1}(t)$ belongs to one of the three orthogonal subspaces of (31), this means that only eigenvectors belonging to $\operatorname{Im}\bar\partial_t$ can have a non-trivial contribution to (32) for $t\neq 0$.

In particular, for every $t\neq 0$, both U_t and V_t are $\bar\partial_t$-exact. We can therefore find, for every $t\neq 0$, a smooth $J_t-(n,n-2)$-form w_t such that $V_t=\bar\partial w_t$. If we choose the form w_t of minimal L^2 norm (with respect to γ_t) with this property, the condition $V_t\in\bigoplus\limits_{\lambda\geq\varepsilon} E^{n,n-1}_{\Delta''_t}(\lambda)$ guarantees that the family of forms $(w_t)_{t\in\Delta^\star}$ extends smoothly across $t=0$ to a family $(w_t)_{t\in\Delta}$ varying in a C^∞ way with t up to $t=0$. This is because the eigenvalues λ contributing to V_t are uniformly bounded below by $\varepsilon>0$ (cf. argument in Section 2).

As for $U_t\in\bigoplus\limits_{\lambda\leq\varepsilon_t} E^{n,n-1}_{\Delta''_t}(\lambda)$, we are unable to guarantee the boundedness near $t=0$ of its $\bar\partial_t$-potential because of the eigenvalues $\lambda_j(t)\leq\varepsilon_t$ converging to 0. Therefore, we will not consider the $\bar\partial_t$-potential. However, the $(n,n-1)$-form U_t is d-closed. Indeed, it is ∂_t-closed in a trivial way for bidegree reasons and is also $\bar\partial_t$-closed (even $\bar\partial_t$-exact, as it has been argued above). Thus, the $\partial\bar\partial$-lemma implies that U_t is d-exact for every $t\neq 0$. We can therefore find, for all $t\neq 0$, a form ξ_t of degree $2n-2$ such that $U_t=d\,\xi_t$. If we choose the form ξ_t of minimal L^2-norm (with respect to γ_t) with this property, we have

$$\xi_t = \Delta_t^{-1}\, d_t^\star U_t, \qquad t\neq 0, \tag{33}$$

where, for all $t \in \Delta$ (including $t = 0$), $\Delta_t = d\,d_t^\star + d_t^\star\,d : C^\infty_{2n-2}(X,\,\mathbb{C}) \rightarrow C^\infty_{2n-2}(X,\,\mathbb{C})$ is the d-Laplacian associated with the metric γ_t and Δ_t^{-1} is the inverse of the restriction of Δ_t to the orthogonal complement of its kernel (the Green operator of Δ_t). Now, the Hodge isomorphism theorem gives :

$$\ker\Delta_t \simeq H^{2n-2}_{DR}(X_t,\mathbb{C}) = H^{2n-2}(X,\mathbb{C}), \qquad t\in\Delta, \tag{34}$$

and we know that all the De Rham cohomology groups $H^{2n-2}_{DR}(X_t,\mathbb{C})$ of the fibres X_t can be identified with a fixed space $H^{2n-2}(X,\mathbb{C})$. In particular, the dimension of $\ker\Delta_t$ is independent of $t\in\Delta$, which means that the positive eigenvalues of Δ_t have a uniform positive (>0) lower bound for t close to 0 (cf. Kodaira-Spencer arguments [Kod86] recalled in Remark 2.3 and applied to the C^∞ family of strongly elliptic operators $(\Delta_t)_{t\in\Delta}$). Thus, in this respect, there is a sharp contrast between the d-Laplacian Δ_t and its $\bar\partial_t$-counterpart Δ''_t : unlike Δ''_t, Δ_t never displays the small eigenvalue phenomenon. In particular, the family of $(2n-2)$-forms $(\xi_t)_{t\in\Delta^\star}$ extends smoothly across $t=0$ to a family $(\xi_t)_{t\in\Delta}$ of forms varying in a C^∞ way with $t\in\Delta$ (up to $t=0$).

Our discussion so far can be summed up as follows.

Lemma 4.2 *Given any family of Gauduchon metrics $(\gamma_t)_{t\in\Delta}$ varying in a C^∞ way with $t\in\Delta$ on the fibres of a family $(X_t)_{t\in\Delta}$ in which the $\partial\bar\partial$-lemma holds on X_t for every $t\neq 0$, we can find a decomposition :*

$$\partial_t\gamma_t^{n-1} = d\,\xi_t + \bar\partial_t w_t, \qquad t\in\Delta, \tag{35}$$

in such a way that

$$d\,\xi_t \in \bigoplus_{\lambda\leq\varepsilon_t} E^{n,n-1}_{\Delta''_t}(\lambda), \qquad \bar\partial_t w_t \in \bigoplus_{\lambda\geq\varepsilon} E^{n,n-1}_{\Delta''_t}(\lambda), \tag{36}$$

where $(w_t)_{t\in\Delta}$ and $(\xi_t)_{t\in\Delta}$ are families of $(2n-2)$-forms and respectively $(n,n-2)$-forms varying in a C^∞ way with $t\in\Delta$ (up to $t=0$), $\varepsilon>0$ is independent of t, $\varepsilon_t>0$ for $t\neq 0$ and ε_t converges to zero as t approaches $0\in\Delta$ (i.e. $\varepsilon_0=0$). Moreover, the following identity holds :

$$\partial_t(\gamma_t^{n-1} - \xi_t^{n-1,\,n-1}) = \bar\partial_t(\xi_t^{n,\,n-2} + w_t), \qquad t\in\Delta. \tag{37}$$

As the form $\xi_t^{n-1,\,n-1}$ need not be real, we find it more convenient to write :

$$\partial_t(\gamma_t^{n-1} - \xi_t^{n-1,\,n-1} - \overline{\xi_t^{n-1,\,n-1}}) = \bar\partial_t(\xi_t^{n,\,n-2} + \overline{\xi_t^{n-2,\,n}} + w_t), \qquad t\in\Delta. \tag{38}$$

To get (37) from (35), it suffices to write $d\,\xi_t = \partial_t\xi_t + \bar\partial_t\xi_t$ and to remember that $d\,\xi_t = U_t$ is a form of pure J_t-type $(n,n-1)$. Hence $d\,\xi_t = \partial_t\xi_t^{n-1,\,n-1} + \bar\partial_t\xi_t^{n,\,n-2}$. The vanishing of the $(n-1,n)$-component of $d\,\xi_t$

amounts to $\bar\partial_t\xi_t^{n-1,\,n-1} + \partial_t\xi_t^{n-2,\,n} = 0$, or equivalently by conjugation to $\partial_t(-\overline{\xi_t^{n-1,\,n-1}}) = \bar\partial_t\overline{\xi_t^{n-2,\,n}}$. Hence (38) follows from (37).

As all the forms involved in (38) vary in a C^∞ way with $t\in\Delta$ (up to $t=0$), to finish the proof of Theorem 1.1 it clearly suffices to show that

$$\gamma_t^{n-1} - \xi_t^{n-1,\,n-1} - \overline{\xi_t^{n-1,\,n-1}} > 0, \quad \text{for all} \quad t\in\Delta. \tag{39}$$

Indeed, if this positivity property has been proved, Michelsohn's observation in linear algebra [Mic83, p. 279-280] enables one to extract the $(n-1)^{st}$ root of $\gamma_t^{n-1} - \xi_t^{n-1,\,n-1} - \overline{\xi_t^{n-1,\,n-1}}$ and to find, for all $t\in\Delta$, a unique $J_t-(1,1)$-form $\rho_t > 0$ such that

$$\gamma_t^{n-1} - \xi_t^{n-1,\,n-1} - \overline{\xi_t^{n-1,\,n-1}} = \rho_t^{n-1}, \quad t\in\Delta. \tag{40}$$

By construction, ρ_t defines a *strongly Gauduchon* metric on X_t for every $t\in\Delta$ thanks to (38). In particular, X_0 is a *strongly Gauduchon* manifold and Proposition 4.1 follows. It actually suffices to prove (39) for $t=0$.

Moreover, it would clearly suffice to prove the stronger property :

$$\xi_0^{n-1,n-1} = 0. \tag{41}$$

If this has been proved, then identity (37) applied to $t=0$ reads $\partial_0\gamma_0^{n-1} = \bar\partial_0(\xi_0^{n,n-2} + w_0)$, hence γ_0 is a *strongly Gauduchon* metric on X_0 and Proposition 4.1 follows.

We have thus reduced our uniform boundedness problem for the *main quantity* I_t to the positivity problem (39) or the vanishing subproblem (41).

The positivity problem

Let $||\cdot|| = ||\cdot||_t$ and $\langle\langle\ ,\ \rangle\rangle = \langle\langle\ ,\ \rangle\rangle_t$ stand for the L^2-norm and respectively the L^2-scalar product defined by the Gauduchon metric γ_t on the forms of X_t.

For the sake of perspicuity, we begin by proving (41) in a special case that brings out the mechanism and locates the difficulty. Different arguments will subsequently be given to settle the positivity problem in full generality.

• *Proof of (41) and implicitly of Proposition 4.1 in an ideal case*

Consider the orthogonal decompositions of γ_t^{n-1} analogous to (32) with respect to the eigenspaces of Δ''_t and respectively Δ'_t acting on J_t-type $(n-1, n-1)$-forms :

$$\gamma_t^{n-1} = u_t + v_t, \quad \text{with } u_t \in \bigoplus_{\mu\leq\delta_t} E_{\Delta''_t}^{n-1,n-1}(\mu), \quad v_t \in \bigoplus_{\mu\geq\delta} E_{\Delta''_t}^{n-1,n-1}(\mu), \quad t\in\Delta, \tag{42}$$

and

$$\gamma_t^{n-1} = \bar{u}_t + \bar{v}_t, \quad \text{with } \bar{u}_t \in \bigoplus_{\mu\leq\delta_t} E_{\Delta'_t}^{n-1,n-1}(\mu), \quad \bar{v}_t \in \bigoplus_{\mu\geq\delta} E_{\Delta'_t}^{n-1,n-1}(\mu), \quad t\in\Delta, \tag{43}$$

where $\mathrm{Spec}\,(\Delta''_t : C^\infty_{n-1,\,n-1}(X_t,\,\mathbb{C}) \to C^\infty_{n-1,\,n-1}(X_t,\,\mathbb{C})) \subset [0,\,\delta_t]\cup[\delta,\,+\infty)$ and $\delta_t \to 0$ as $t\to 0$, while $\delta>0$ is independent of t. The rest of the notation is analogous to that used earlier for $(n, n-1)$-forms, the symbol $E^{p,q}(\lambda)$ denoting eigenspaces of (p,q)-forms with eigenvalue λ. Decompositions (42) and (43) are conjugate to each other because γ_t^{n-1} is a real form and $\overline{\Delta'_t} = \overline{\partial_t\partial_t^\star + \partial_t^\star\partial_t} = \bar\partial_t\bar\partial_t^\star + \bar\partial_t^\star\bar\partial_t = \Delta''_t$. In particular, the eigenvalues of both Δ'_t and Δ''_t being real (even non-negative, by self-adjointness), the equivalence holds : $u\in E_{\Delta''_t}^{n-1,n-1}(\lambda) \Leftrightarrow \bar{u}\in E_{\Delta'_t}^{n-1,n-1}(\lambda)$.

Definition 4.3 *We say that the* **ideal case** *occurs if*

(i) $u_t = \bar{u}_t$ *for all* $t\in\Delta$. *In other words, the forms* u_t *and* v_t *into which* γ_t^{n-1} *splits in (42) are real;*

(ii) $\partial_t\Delta''_t u = \Delta''_t\partial_t u$ *for all* $u\in C^\infty_{n-1,\,n-1}(X_t,\mathbb{C})$ *and all* $t\in\Delta$. *In other words,* ∂_t *commutes with* Δ''_t *on* J_t*-type* $(n-1, n-1)$*-forms.*

If the Laplacians Δ'_t and Δ''_t were calculated with respect to a *Kähler metric*, then $\Delta'_t = \Delta''_t$ and the *ideal case* would occur since ∂_t always commutes with Δ'_t. The failure of the *ideal case* to occur in general is caused by the failure of the Gauduchon metric γ_t to be Kähler.

Lemma 4.4 *Let* $(X_t)_{t\in\Delta}$ *be any family such that the* $\partial\bar\partial$*-lemma holds on* X_t *for every* $t\neq 0$. *Let* $(\gamma_t)_{t\in\Delta}$ *be any family of Gauduchon metrics varying in a* C^∞ *way with* $t\in\Delta$ *on the fibres* $(X_t)_{t\in\Delta}$. *Suppose the* **ideal case** *occurs. The notation being that of Lemma 4.2, the following estimate holds :*

$$||\xi_t^{n-1,\,n-1}|| \leq \varepsilon_0(t)\,||\gamma_t^{n-1}||, \quad \text{for all } t\in\Delta\setminus\{0\}, \tag{44}$$

with a constant $\varepsilon_0(t)>0$ *converging to zero as* $t\to 0$. *In particular,* $\xi_0^{n-1,n-1} = 0$ *and the metric* γ_0 *is strongly Gauduchon, proving Proposition 4.1.*

To infer the second statement from estimate (44), it suffices to remember that $(\gamma_t^{n-1})_{t\in\Delta}$ and the norms $(||\cdot|| = ||\cdot||_t)_{t\in\Delta}$ vary in a C^∞ way with t (up to $t=0$). This clearly implies that $||\gamma_t^{n-1}||$ has a positive upper bound independent of t if t is close to 0 (it actually converges to $||\gamma_0^{n-1}||\in(0,+\infty)$ when $t\to 0$). Hence $\xi_0^{n-1,n-1}=0$, i.e. (41).

Proof of Lemma 4.4. Condition (ii) of Definition 4.3 implies that the decomposition (35) of $\partial_t\gamma_t^{n-1}$ (which is known to satisfy (36) and to be unique with

this property) is obtained by applying ∂_t on both sides of decomposition (42) of γ_t^{n-1}. Thus $\varepsilon_t = \delta_t$, $\varepsilon = \delta$ and

$$\partial_t u_t = d\,\xi_t, \quad t \in \Delta. \tag{45}$$

On the other hand, property (i) of Definition 4.3, combined with (43), gives :

$$u_t = \bar{u}_t \in \bigoplus_{\mu \leq \delta_t = \varepsilon_t} E^{n-1,n-1}_{\Delta'_t}(\mu), \quad t \in \Delta. \tag{46}$$

As explained earlier, the positive eigenvalues of the d-Laplacian Δ_t : $C^\infty_{2n-2}(X,\,\mathbb{C}) \rightarrow C^\infty_{2n-2}(X,\,\mathbb{C})$ defined by the metric γ_t have a positive lower bound independent of t if $t \in \Delta$ is close to 0. This means that there exists a constant $c > 0$, independent of $t \in \Delta$, such that the restriction of Δ_t to the orthogonal complement of its kernel satisfies

$$(\Delta_t)_{|(\ker \Delta_t)^\perp} \geq c\,\mathrm{Id}, \qquad t \in \Delta, \tag{47}$$

after possibly shrinking the base Δ about 0. Putting the bits together, we get the following estimate :

$$\begin{aligned} c\,||\xi_t^{n-1,\,n-1}||^2 \;&\leq\; c\,||\xi_t||^2 \leq \langle\langle \Delta_t \xi_t,\, \xi_t \rangle\rangle = ||d\,\xi_t||^2 \\ &=\; ||\partial_t u_t||^2 \leq \langle\langle \Delta'_t u_t,\, u_t\rangle\rangle \leq \varepsilon_t\,||u_t||^2 \\ &\leq\; \varepsilon_t\,||\gamma_t^{n-1}||^2, \qquad \text{for all } t \in \Delta \setminus \{0\}. \end{aligned} \tag{48}$$

Indeed, on the first line : the first inequality follows from the components $\xi_t^{n,\,n-2}$, $\xi_t^{n-1,\,n-1}$, $\xi_t^{n-2,\,n}$, that split ξ_t into J_t-types, being mutually orthogonal as forms of different pure types ; the second inequality follows from (47) as ξ_t has been chosen of minimal L^2-norm in (33), hence $\xi_t \in \mathrm{Im}\, d_t^\star$, so, in particular, $\xi_t \in (\ker \Delta_t)^\perp$ and (47) applies ; the identity follows from $d_t^\star \xi_t = 0$ which holds because $\mathrm{Im}\, d_t^\star \subset \ker d_t^\star$. Further down on the second line : the equality with the last term of the first line follows from (45) ; the first inequality is obvious as $\langle\langle \Delta'_t u_t,\, u_t\rangle\rangle = ||\partial_t u_t||^2 + ||\partial_t^\star u_t||^2$; the second inequality follows from (46). Finally, the inequality between the last term of the second line and the term on the third line is obvious from the decomposition (42) being orthogonal.

The conclusion of (48) is that

$$||\xi_t^{n-1,\,n-1}||^2 \leq \frac{\varepsilon_t}{c}\,||\gamma_t^{n-1}||^2, \qquad \text{for all } t \in \Delta \setminus \{0\}, \tag{49}$$

which is nothing but estimate (44) with $\varepsilon_0(t) := \frac{\varepsilon_t}{c} \rightarrow 0$ as $t \rightarrow 0$ that we had set out to prove. This concludes the proof of Lemma 4.4. $\square$

Notice that if we disregard estimate (44), the weaker conclusion $d\,\xi_0 = 0$, which suffices for our purposes since it gives $\partial_0 \gamma_0^{n-1} = \bar\partial_0 w_0$ hence γ_0 is *strongly Gauduchon*, can be reached by a quicker route. Indeed, by (36), $d\,\xi_t$

lies in the Δ''_t-eigenspaces with eigenvalues $\lambda \leq \epsilon_t \to 0$ as $t \to 0$. Hence $d\,\xi_0$ is Δ''_0-harmonic (or, equivalently, both $\bar\partial_0$ and $\bar\partial_0^\star$-closed). Moreover, if the *ideal case* occurs, (45) and (46) show that $d\,\xi_t$ has the similar property with respect to Δ'_t, hence $d\,\xi_0$ is also Δ'_0-harmonic (or, equivalently, both ∂_0 and $\partial_0^\star$-closed). Now, since $d\,\xi_0$ is of pure type and harmonic for both Δ'_0 and Δ''_0, it must be Δ_0-harmonic (i.e. both d and $d_0^\star$-closed) since $d = \partial_0 + \bar\partial_0$ and $d_0^\star = \partial_0^\star + \bar\partial_0^\star$. As $d\,\xi_0$ is obviously d-exact and as the spaces $\ker \Delta_0$ and $\mathrm{Im}\, d$ are orthogonal, we must have $d\,\xi_0 = 0$.

- *Proof of Proposition 4.1 in the general case*

The ideal thing would be proving the desired positivity property (39) right away. As $\gamma_t^{n-1} > 0$, it suffices to show that the L^2-norm $||\cdot||$ of $\xi_t^{n-1,\,n-1}$ can be made arbitrarily small (hence so can the L^2-norm of the real form $\xi_t^{n-1,\,n-1} + \overline{\xi_t^{n-1,\,n-1}}$) uniformly w.r.t. $t \in \Delta$. (It would suffice to guarantee this property when $t = 0$.) Indeed, in that case, $\gamma_t^{n-1} - \xi_t^{n-1,\,n-1} - \overline{\xi_t^{n-1,\,n-1}}$ would be $\partial_t\bar\partial_t$-cohomologous to an $(n-1,\,n-1)$-form $\rho_t^{n-1} > 0$ for all $t \in \Delta$. This is because the Bott-Chern cohomology groups $H_{BC}^{n-1,\,n-1}(X_t,\,\mathbb{C})$ can be calculated using either C^∞-forms, or L^2-forms, or currents or indeed forms of other regularity. Such a form $\rho_0^{n-1} > 0$ would induce a *strongly Gauduchon* metric $\rho_0 > 0$ on X_0 after extracting Michelsohn's $(n-1)^{st}$ root. However, we can see no reason why the norm of $\xi_t^{n-1,\,n-1}$ should be as small as needed if the *ideal case* does not occur. The way out of this difficulty is to iterate the construction described in Lemma 4.2 so that $||\xi_t^{n-1,\,n-1}||$ becomes arbitrarily small after a sufficient number of iterations.

The first observation is that the $\partial\bar\partial$-lemma allows us to iterate Lemma 4.2 indefinitely. Identities (50) below compare to (37) and (51) to (38).

Lemma 4.5 *For every $p \in \mathbb{N}$, there exist families $(\Omega_{t,\,(p)}^{n-1,\,n-1})_{t\in\Delta}$ of $J_t-(n-1,\,n-1)$-forms and $(\xi_{t,\,(p)})_{t\in\Delta}$ of $(2n-2)$-forms varying in a C^∞ way with t (up to $t=0$) such that, for all $t \in \Delta$, we have :*

$$\begin{aligned}\partial_t(\gamma_t^{n-1} - \Omega_{t,\,(p)}^{n-1,\,n-1}) &= \partial_t(\gamma_t^{n-1} - \xi_{t,\,(p)}^{n-1,\,n-1}) \\ &= \bar\partial_t(\xi_{t,\,(p)}^{n,\,n-2} + \xi_{t,\,(p-1)}^{n,\,n-2} + \cdots + \xi_{t,\,(1)}^{n,\,n-2} + \xi_t^{n,\,n-2} + w_t),\end{aligned} \tag{50}$$

where, as usual, $\xi_{t,\,(l)}^{r,\,s}$ denotes the component of J_t-type $(r,\,s)$ of $\xi_{t,\,(l)}$. As the form $\xi_{t,\,(p)}^{n-1,\,n-1}$ need not be real, we find it more convenient to write :

$$\partial_t(\gamma_t^{n-1} - \xi_{t,\,(p)}^{n-1,\,n-1} - \overline{\xi_{t,\,(p)}^{n-1,\,n-1}}) = \bar\partial_t(\xi_{t,\,(p)}^{n,\,n-2} + \overline{\xi_{t,\,(p)}^{n-2,\,n}} + \xi_{t,\,(p-1)}^{n,\,n-2} + \cdots + \xi_t^{n,\,n-2} + w_t). \tag{51}$$

Proof. We have already noticed that $\partial_t\gamma_t^{n-1}$ and its projections $d\,\xi_t$ and $\bar\partial_t w_t$ given in (35) are all d, ∂_t and $\bar\partial_t$-exact for all $t \neq 0$. Writing $d\,\xi_t = \partial_t\xi_t^{n-1,\,n-1} +$

$\bar\partial_t\xi_t^{n,\,n-2}$, we see that $\bar\partial_t\xi_t^{n,\,n-2}$ is $\bar\partial_t$-closed (even $\bar\partial_t$-exact) and is also ∂_t-closed for bidegree reasons (being of pure type $(n, n-1)$). Thus $\bar\partial_t\xi_t^{n,\,n-2}$ is d-closed and of pure type. By the $\partial\bar\partial$-lemma, the $\bar\partial_t$-exactness of $\bar\partial_t\xi_t^{n,\,n-2}$ implies its d and ∂_t-exactness for all $t\neq 0$. Then $\partial_t\xi_t^{n-1,\,n-1}$ must also be d and ∂_t-exact for all $t\neq 0$ as a difference of two such forms. We can thus write

$$\partial_t\xi_t^{n-1,\,n-1} = \partial_t\Omega_t^{n-1,\,n-1} = d\,\xi_{t,\,(1)}, \quad t\in\Delta, \tag{52}$$

where $\Omega_t^{n-1,\,n-1}$ stands for the ∂_t-potential of minimal L^2-norm $||\cdot||$ and $\xi_{t,\,(1)}$ denotes the d-potential of minimal L^2-norm $||\cdot||$ of $\partial_t\xi_t^{n-1,\,n-1}$. Identities (52) *a priori* hold only for $t\neq 0$ as the $\partial\bar\partial$-lemma is only known to apply on X_t with $t\neq 0$. However, we have seen that the limit on X_0, when it exists, of any family of forms that are d-exact on X_t for all $t\neq 0$ is still d-exact on X_0 owing to the De Rham cohomology being constant on the fibres X_t, $t\in\Delta$ (no *small eigenvalue* phenomenon for Δ_t). Thus $\partial_0\xi_0^{n-1,\,n-1}$ is d-exact, hence the family $(\xi_{t,\,(1)})_{t\in\Delta}$ is defined up to $t=0$. Meanwhile, $||\Omega_t^{n-1,\,n-1}||\leq||\xi_t^{n-1,\,n-1}||$ for all $t\in\Delta^\star$ by the L^2-norm minimality of $\Omega_t^{n-1,\,n-1}$. As $\xi_t^{n-1,\,n-1}$ is known to extend in a C^∞ way to X_0, so must $\Omega_t^{n-1,\,n-1}$. Thus identities (52) hold for all $t\in\Delta$ (including $t=0$) and the families $(\Omega_t^{n-1,\,n-1})_{t\in\Delta}$ and $(\xi_{t,\,(1)})_{t\in\Delta}$ vary in a C^∞ way with t.

In view of (52), identity (37) becomes :

$$\partial_t(\gamma_t^{n-1}-\Omega_t^{n-1,\,n-1}) = \partial_t(\gamma_t^{n-1}-\xi_t^{n-1,\,n-1}) = \bar\partial_t(\xi_t^{n,\,n-2}+w_t), \quad t\in\Delta. \tag{53}$$

Writing $d\,\xi_{t,\,(1)} = \partial_t\xi_{t,\,(1)}^{n-1,\,n-1}+\bar\partial_t\xi_{t,\,(1)}^{n,\,n-2}$ (recall that $d\,\xi_{t,\,(1)}$ is of J_t-type $(n,\,n-1)$) and using (52), we get :

$$\partial_t(\gamma_t^{n-1}-\xi_{t,\,(1)}^{n-1,\,n-1}) = \bar\partial_t(\xi_{t,\,(1)}^{n,\,n-2}+\xi_t^{n,\,n-2}+w_t), \quad t\in\Delta. \tag{54}$$

The procedure described above can now be iterated indefinitely. The right-hand term in (54) is a d-closed and $\bar\partial_t$-exact $(n, n-1)$-form, hence it must be d, ∂_t and $\bar\partial_t$-exact for all $t\neq 0$ by the $\partial\bar\partial$-lemma. Then so is $\partial_t\xi_{t,\,(1)}^{n-1,\,n-1}$ as a difference of two such forms (i.e. $\partial_t\gamma_t^{n-1}$ and the right-hand term in (54)). We then get identities analogous to (52) :

$$\partial_t\xi_{t,\,(1)}^{n-1,\,n-1} = \partial_t\Omega_{t,\,(1)}^{n-1,\,n-1} = d\,\xi_{t,\,(2)}, \quad t\in\Delta,$$

where $\Omega_{t,\,(1)}^{n-1,\,n-1}$ and $\xi_{t,\,(2)}$ are the ∂_t and respectively d-potentials of $\partial_t\xi_{t,\,(1)}^{n-1,\,n-1}$ with minimal L^2-norms. They extend smoothly to X_0 by the same arguments as above and, writing $d\,\xi_{t,\,(2)} = \partial_t\xi_{t,\,(2)}^{n-1,\,n-1}+\bar\partial_t\xi_{t,\,(2)}^{n,\,n-2}$, (54) reads :

$$\partial_t(\gamma_t^{n-1}-\xi_{t,\,(2)}^{n-1,\,n-1}) = \bar\partial_t(\xi_{t,\,(2)}^{n,\,n-2}+\xi_{t,\,(1)}^{n,\,n-2}+\xi_t^{n,\,n-2}+w_t), \quad t\in\Delta. \tag{55}$$

The $(n,\,n-1)$-form $\partial_t\xi_{t,\,(2)}^{n-1,\,n-1}$ is again d, ∂_t and $\bar\partial_t$-exact for all $t\neq 0$ by the $\partial\bar\partial$-lemma and the procedure can be repeated. At step p one gets :

$$\partial_t\xi_{t,(p)}^{n-1,\,n-1} = \partial_t\Omega_{t,(p)}^{n-1,\,n-1} = d\,\xi_{t,(p+1)}, \quad t\in\Delta,\ p\in\mathbb{N}, \tag{56}$$

with $\Omega_{t,(p)}^{n-1,\,n-1}$ and $\xi_{t,(p+1)}$ the ∂_t and respectively d-potentials of minimal L^2-norms of $\partial_t\xi_{t,(p)}^{n-1,\,n-1}$. The form $\Omega_{t,(p)}^{n-1,\,n-1}$ can be seen as a correction of $\xi_{t,(p)}^{n-1,\,n-1}$ if the latter does not have minimal L^2-norm. It is clear that the analogue for p of (53), (54), (55) and the definition of $\Omega_{t,(p)}^{n-1,\,n-1}$ in (56) add up to the identities (50) claimed in the statement. To get (51) from (50), recall that $\partial_t\xi_{t,(p-1)}^{n-1,\,n-1} = d\,\xi_{t,(p)}$ is of J_t-type $(n,\,n-1)$, hence its $(n-1,\,n)$-component $\partial_t\xi_{t,(p)}^{n-2,\,n} + \bar\partial_t\xi_{t,(p)}^{n-1,\,n-1}$ vanishes. Taking conjugates, one gets $\partial_t(-\overline{\xi_{t,(p)}^{n-1,\,n-1}}) = \bar\partial_t\overline{\xi_{t,(p)}^{n-2,\,n}}$ and this term can be added to (50) to get (51). $\square$

For a technical reason that will become apparent subsequently, the norms of the forms $\xi_{t,(p)}^{n-1,\,n-1}$ can be more easily estimated if the inductive construction described in Lemma 4.5 above is slightly altered in the following way.

Step 1 of the new inductive construction. By (37) of Lemma 4.2 we get

$$\partial_t\gamma_t^{n-1} = \partial_t\xi_t^{n-1,\,n-1} + \bar\partial_t(\xi_t^{n,\,n-2} + w_t), \quad t\in\Delta. \tag{57}$$

Let $(\eta_t)_{t\in\Delta}$ be a smooth family of nonzero J_t-$(n,\,n-1)$-forms satisfying the following three conditions $(\star)$:

(a) $\eta_t = \partial_t\nu_t^{n-1,\,n-1} = \bar\partial_t\vartheta_t^{n,\,n-2}$ for all $t\in\Delta$ and for smooth families of forms $(\nu_t^{n-1,\,n-1})_{t\in\Delta}$, $(\vartheta_t^{n,\,n-2})_{t\in\Delta}$ of the shown types;

(b) $||\xi_t^{n-1,\,n-1} + \nu_t^{n-1,\,n-1}|| \leq ||\xi_t^{n-1,\,n-1}||, \quad t\in\Delta;$

(c) for all $t\in\Delta$ and for some $\varepsilon_0>0$ independent of t we have

$$\frac{\langle\langle\Delta''_t(\partial_t\xi_t^{n-1,\,n-1} + \partial_t\nu_t^{n-1,\,n-1}),\,\partial_t\xi_t^{n-1,\,n-1} + \partial_t\nu_t^{n-1,\,n-1}\rangle\rangle}{\langle\langle\Delta'_t(\partial_t\xi_t^{n-1,\,n-1} + \partial_t\nu_t^{n-1,\,n-1}),\,\partial_t\xi_t^{n-1,\,n-1} + \partial_t\nu_t^{n-1,\,n-1}\rangle\rangle} \geq \varepsilon_0 > 0.$$

Now using (a), (57) becomes :

$$\partial_t\gamma_t^{n-1} = \partial_t(\xi_t^{n-1,\,n-1} + \nu_t^{n-1,\,n-1}) + \bar\partial_t(\xi_t^{n,\,n-2} + w_t - \vartheta_t^{n,\,n-2}), \quad t\in\Delta. \tag{58}$$

Let $\Omega_t^{n-1,\,n-1}$ and $\widetilde\xi_{t,(1)}$ be the ∂_t-potential and respectively the d-potential of minimal L^2-norms of $\partial_t(\xi_t^{n-1,\,n-1} + \nu_t^{n-1,\,n-1})$:

$$\partial_t(\xi_t^{n-1,\,n-1} + \nu_t^{n-1,\,n-1}) = \partial_t\Omega_t^{n-1,\,n-1} = d\,\widetilde\xi_{t,(1)}, \quad t\in\Delta. \tag{59}$$

Notice that, since $d\,\widetilde\xi_{t,(1)}$ is of pure type $(n,\,n-1)$, we must have

$$d\,\widetilde\xi_{t,(1)} = \partial_t\widetilde\xi_{t,(1)}^{n-1,\,n-1} + \bar\partial_t\widetilde\xi_{t,(1)}^{n,\,n-2}, \quad t\in\Delta.$$

Thus (58) now reads :

$$\partial_t\gamma_t^{n-1} = \partial_t\widetilde{\xi}_{t,\,(1)}^{n-1,\,n-1} + \bar\partial_t(\widetilde{\xi}_{t,\,(1)}^{n,\,n-2} + \xi_t^{n,\,n-2} + w_t - \vartheta_t^{n,\,n-2}), \quad t\in\Delta. \tag{60}$$

Step $p+1$ of the new inductive construction. Suppose that Step p has been performed and has produced the following decomposition for all $t\in\Delta$:

$$\begin{aligned}\partial_t\gamma_t^{n-1} = \partial_t\widetilde{\xi}_{t,\,(p)}^{n-1,\,n-1} & + \bar\partial_t(\widetilde{\xi}_{t,\,(p)}^{n,\,n-2} + \cdots + \widetilde{\xi}_{t,\,(1)}^{n,\,n-2} + \xi_t^{n,\,n-2} + w_t \\ & - \vartheta_t^{n,\,n-2} - \vartheta_{t,\,(1)}^{n,\,n-2} - \cdots - \vartheta_{t,\,(p-1)}^{n,\,n-2}).\end{aligned} \tag{61}$$

Let $(\eta_{t,\,(p)})_{t\in\Delta}$ be a smooth family of nonzero J_t-$(n,\,n-1)$-forms satisfying the following three conditions $(\star_p)$:

(a) $\eta_{t,\,(p)} = \partial_t\nu_{t,\,(p)}^{n-1,\,n-1} = \bar\partial_t\vartheta_{t,\,(p)}^{n,\,n-2}$ for all $t\in\Delta$ and for smooth families of forms $(\nu_{t,\,(p)}^{n-1,\,n-1})_{t\in\Delta}$, $(\vartheta_{t,\,(p)}^{n,\,n-2})_{t\in\Delta}$ of the shown types;

(b) $||\widetilde{\xi}_{t,\,(p)}^{n-1,\,n-1} + \nu_{t,(p)}^{n-1,\,n-1}|| \leq ||\widetilde{\xi}_{t,\,(p)}^{n-1,\,n-1}||, \quad t\in\Delta;$

(c) for all $t\in\Delta$ and for some $\varepsilon_0>0$ independent of t and of $p\in\mathbb{N}^\star$ we have

$$\frac{\langle\langle\Delta''_t(\partial_t\widetilde{\xi}_{t,\,(p)}^{n-1,\,n-1} + \partial_t\nu_{t,\,(p)}^{n-1,\,n-1}),\, \partial_t\widetilde{\xi}_{t,\,(p)}^{n-1,\,n-1} + \partial_t\nu_{t,\,(p)}^{n-1,\,n-1}\rangle\rangle}{\langle\langle\Delta'_t(\partial_t\widetilde{\xi}_{t,\,(p)}^{n-1,\,n-1} + \partial_t\nu_{t,\,(p)}^{n-1,\,n-1}),\, \partial_t\widetilde{\xi}_{t,\,(p)}^{n-1,\,n-1} + \partial_t\nu_{t,\,(p)}^{n-1,\,n-1}\rangle\rangle} \geq \varepsilon_0 > 0.$$

Now using (a), (61) becomes for all $t\in\Delta$:

$$\begin{aligned}\partial_t\gamma_t^{n-1} = \partial_t(\widetilde{\xi}_{t,\,(p)}^{n-1,\,n-1} + \nu_{t,\,(p)}^{n-1,\,n-1}) & + \bar\partial_t(\widetilde{\xi}_{t,\,(p)}^{n,\,n-2} + \cdots + \widetilde{\xi}_{t,\,(1)}^{n,\,n-2} + \xi_t^{n,\,n-2} + w_t \\ & - \vartheta_t^{n,\,n-2} - \vartheta_{t,\,(1)}^{n,\,n-2} - \cdots - \vartheta_{t,\,(p)}^{n,\,n-2}).\end{aligned} \tag{62}$$

Let $\Omega_{t,\,(p)}^{n-1,\,n-1}$ and $\widetilde{\xi}_{t,\,(p+1)}$ be the ∂_t-potential and respectively the d-potential of minimal L^2-norms of $\partial_t(\widetilde{\xi}_{t,\,(p)}^{n-1,\,n-1} + \nu_{t,\,(p)}^{n-1,\,n-1})$:

$$\partial_t(\widetilde{\xi}_{t,\,(p)}^{n-1,\,n-1} + \nu_{t,\,(p)}^{n-1,\,n-1}) = \partial_t\Omega_{t,\,(p)}^{n-1,\,n-1} = d\,\widetilde{\xi}_{t,\,(p+1)}, \quad t\in\Delta. \tag{63}$$

Notice that, since $d\,\widetilde{\xi}_{t,\,(p+1)}$ is of pure type $(n,\,n-1)$, we must have

$$d\,\widetilde{\xi}_{t,\,(p+1)} = \partial_t\widetilde{\xi}_{t,\,(p+1)}^{n-1,\,n-1} + \bar\partial_t\widetilde{\xi}_{t,\,(p+1)}^{n,\,n-2}, \quad t\in\Delta.$$

Thus (62) now reads for all $t\in\Delta$:

$$\begin{aligned}\partial_t\gamma_t^{n-1} = \partial_t\widetilde{\xi}_{t,\,(p+1)}^{n-1,\,n-1} & + \bar\partial_t(\widetilde{\xi}_{t,\,(p+1)}^{n,\,n-2} + \cdots + \widetilde{\xi}_{t,\,(1)}^{n,\,n-2} + \xi_t^{n,\,n-2} + w_t \\ & - \vartheta_t^{n,\,n-2} - \vartheta_{t,\,(1)}^{n,\,n-2} - \cdots - \vartheta_{t,\,(p)}^{n,\,n-2}),\end{aligned} \tag{64}$$

completing the inductive construction of the families $(\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)})_{t\in\Delta}$, $p\in\mathbb{N}$. (We have set $\widetilde{\xi}^{n-1,\,n-1}_{t,\,(0)} := \xi^{n-1,\,n-1}_t$ as well as $\Omega^{n-1,\,n-1}_{t,\,(0)} := \Omega^{n-1,\,n-1}_t$ and $\nu^{n-1,\,n-1}_{t,\,(0)} := \nu^{n-1,\,n-1}_t$ to unify the notation.)

A word of explanation is in order to account for the existence of smooth families $(\eta_{t,\,(p)})_{t\in\Delta}$ satisfying conditions $(\star_p)$ for all $p\in\mathbb{N}$. Let $(\eta_{t,\,(p)})_{t\in\Delta}$ be a smooth family of nonzero J_t-$(n,\,n-1)$-forms so that $\eta_{t,\,(p)}$ is $\bar\partial_t$-exact and

$$\eta_{t,\,(p)} \in \bigoplus_{\mu\geq\varepsilon'} E^{n,\,n-1}_{\Delta'_t}(\mu) \cap \bigoplus_{\lambda\geq\varepsilon''} E^{n,\,n-1}_{\Delta''_t}(\lambda), \quad t\in\Delta, \tag{65}$$

where $\varepsilon',\,\varepsilon''>0$ are uniform lower bounds for the parts of the spectra of Δ'_t and respectively Δ''_t that do not tend to zero as t approaches $0\in\Delta$. As only at most finitely many, if any, positive eigenvalues of Δ'_t and respectively Δ''_t lie below ε' and respectively ε'', the above direct sums of eigenspaces are each of finite codimension in $C^\infty_{n,\,n-1}(X_t,\,\mathbb{C})$, hence such an $\eta_{t,\,(p)}$ can be found. Being of type $(n,\,n-1)$, $\eta_{t,\,(p)}$ is automatically ∂_t-closed, hence also d-closed by the $\bar\partial_t$-exactness assumption. It is then ∂_t-exact for all $t\neq 0$ by the $\partial\bar\partial$-lemma. Since $\eta_{t,\,(p)}$ avoids the *small* eigenvalues of Δ'_t by definition, $\eta_{0,\,(p)}$ is still ∂_0-exact. Similarly, $\eta_{0,\,(p)}$ is also $\bar\partial_0$-exact. Thus we can write $\eta_{t,\,(p)}$ as in (a) of $(\star_p)$. While this construction is so far independent of $p\in\mathbb{N}$, we can now adapt the forms $\eta_{t,\,(p)}$ to match the previously defined forms $\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)}$ and achieve conditions (b) and (c) of $(\star_p)$. Indeed, for every $t\in\Delta$ and every $p\in\mathbb{N}$, there exists an open subset $\mathcal{U}_{t,\,(p)}\subset C^\infty_{n-1,\,n-1}(X_t,\,\mathbb{C})$ such that if $\nu^{n-1,\,n-1}_{t,\,(p)}$ is any form in $\mathcal{U}_{t,\,(p)}$ then either $||\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)} + \nu^{n-1,\,n-1}_{t,\,(p)}|| \leq ||\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)}||$ or $||\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)} - \nu^{n-1,\,n-1}_{t,\,(p)}|| \leq ||\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)}||$. This enables one to achieve (b) simultaneously with (a) (replace $\eta_{t,(p)}$ with $-\eta_{t,\,(p)}$ if necessary). Now (c) is guaranteed whenever the distance from $\partial_t\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)} + \partial_t\nu^{n-1,\,n-1}_{t,\,(p)}$ to $\ker\Delta''_t$ is bounded below by a positive constant independent of t and p (a condition that can be achieved by rescaling each $\partial_t\nu^{n-1,\,n-1}_{t,\,(p)}$ with a positive factor $\delta'_p>0$ independent of $t\in\Delta$ thanks to $\partial_t\nu^{n-1,\,n-1}_{t,\,(p)}$ being $\bar\partial_t$-exact for all $t\in\Delta$ and to $\ker\Delta''_t\perp\operatorname{Im}\bar\partial_t$) if, as $p\to+\infty$, the possible growth towards $+\infty$ of the (finitely many) eigenvalues of Δ'_t for which $\partial_t\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)} + \partial_t\nu^{n-1,\,n-1}_{t,\,(p)}$ has a nontrivial orthogonal projection onto the corresponding Δ'_t-eigenspace is offset by eigenvalues of Δ''_t growing to $+\infty$ with the similar property. The latter condition can be achieved since, by the finite dimensionality of the eigenspaces of Δ'_t and Δ''_t, for any $A>0$ there exists $B>0$ independent of t such that $\bigoplus\limits_{\mu\leq A} E^{n,\,n-1}_{\Delta'_t}(\mu)\subset\bigoplus\limits_{\lambda\leq B} E^{n,\,n-1}_{\Delta''_t}(\lambda)$ for $t\in\Delta$ close to 0.

With these new definitions in place, the identities of Lemma 4.5 are transformed as follows.

Lemma 4.6 *The family $(\widetilde{\xi}_{t,\,(p)})_{t\in\Delta}$ of $(2n-2)$-forms constructed above varies in a C^∞ way with t (up to $t=0$) and satisfies for all $t\in\Delta$ and all $p\in\mathbb{N}$:*

$$\partial_t(\gamma_t^{n-1} - \widetilde{\xi}_{t,(p)}^{n-1,n-1} - \overline{\widetilde{\xi}_{t,(p)}^{n-1,n-1}}) = \bar\partial_t(\widetilde{\xi}_{t,(p)}^{n,n-2} + \overline{\widetilde{\xi}_{t,(p)}^{n-2,n}} + \cdots + \widetilde{\xi}_{t,(1)}^{n,n-2} + \xi_t^{n,n-2}$$
$$+ w_t - \vartheta_t^{n,n-2} - \vartheta_{t,(1)}^{n,n-2} - \cdots - \vartheta_{t,(p-1)}^{n,n-2}). \tag{66}$$

Proof. It follows trivially from (64) with $p+1$ replaced by p and the fact that $d\,\widetilde{\xi}_{t,(p)} = \partial_t\widetilde{\xi}_{t,(p)}^{n-1,n-1} + \bar\partial_t\widetilde{\xi}_{t,(p)}^{n,n-2}$ is of type $(n,\,n-1)$ (thus its $(n-1,\,n)$-component vanishes, hence $-\bar\partial_t\widetilde{\xi}_{t,(p)}^{n-1,n-1} = \partial_t\widetilde{\xi}_{t,(p)}^{n-2,n}$ and taking conjugates $-\partial_t\overline{\widetilde{\xi}_{t,(p)}^{n-1,n-1}} = \bar\partial_t\overline{\widetilde{\xi}_{t,(p)}^{n-2,n}}$) by arguments analogous to those of Lemma 4.5. $\Box$

The next, more substantial step is to show that the L^2-norm of $\widetilde{\xi}_{t,(p)}^{n-1,n-1}$ decreases strictly at each step p of the above inductive construction in a way that guarantees it to become arbitrarily small when p becomes large enough. The following lemma and its corollary provide the final argument to the proof of Proposition 4.1 and, implicitly, to that of Theorem 1.1.

Lemma 4.7 *There exists $\varepsilon > 0$ independent of $t \in \Delta$ and of $p \in \mathbb{N}$ such that the minimal L^2-norm solutions $\Omega_{t,(p)}^{n-1,n-1}$ and $\widetilde{\xi}_{t,(p+1)}$ of the equations*

$$\partial_t\Omega_{t,(p)}^{n-1,n-1} = \partial_t(\widetilde{\xi}_{t,(p)}^{n-1,n-1} + \nu_{t,(p)}^{n-1,n-1}) \quad \text{and} \quad d\,\widetilde{\xi}_{t,(p+1)} = \partial_t(\widetilde{\xi}_{t,(p)}^{n-1,n-1} + \nu_{t,(p)}^{n-1,n-1}) \tag{67}$$

satisfy the L^2-norm estimates :

$$||\widetilde{\xi}_{t,(p+1)}|| \leq \frac{1}{\sqrt{1+\varepsilon}}\,||\Omega_{t,(p)}^{n-1,n-1}||, \quad t \in \Delta,\, p \in \mathbb{N}. \tag{68}$$

Before proving this statement, we notice an immediate corollary.

Corollary 4.8 *The form $\widetilde{\xi}_{t,(p)}^{n-1,n-1}$ obtained at step p above satisfies*

$$||\widetilde{\xi}_{t,(p)}^{n-1,n-1}|| \leq \frac{1}{(\sqrt{1+\varepsilon})^p}\,||\xi_t^{n-1,n-1}||, \quad t \in \Delta,\; p \in \mathbb{N}. \tag{69}$$

In particular, $||\widetilde{\xi}_{t,(p)}^{n-1,n-1}||$ can be made arbitrarily small, uniformly in $t \in \Delta$, if the number p of iterations of the inductive procedure is sufficiently large.

Proof of Corollary 4.8. Assuming that Lemma 4.7 has been proved, we get :

$$||\widetilde{\xi}_{t,(p+1)}|| \leq \frac{1}{\sqrt{1+\varepsilon}}\,||\Omega_{t,(p)}^{n-1,n-1}|| \leq \frac{1}{\sqrt{1+\varepsilon}}\,||\widetilde{\xi}_{t,(p)}^{n-1,n-1} + \nu_{t,(p)}^{n-1,n-1}||, \quad p \in \mathbb{N}.$$

The latter inequality follows from the L^2-norm minimality of $\Omega_{t,(p)}^{n-1,n-1}$ among the solutions of the equation $\partial_t\Omega_{t,(p)}^{n-1,n-1} = \partial_t(\widetilde{\xi}_{t,(p)}^{n-1,n-1} + \nu_{t,(p)}^{n-1,n-1})$. We

also have $||\widetilde{\xi}_{t,\,(p+1)}^{n-1,\,n-1}|| \leq ||\widetilde{\xi}_{t,\,(p+1)}||$ since the former form is the $(n-1,\,n-1)$-component of the latter and forms of distinct pure types are orthogonal. Combining with (b) of properties $(\star_p)$, we get

$$||\widetilde{\xi}_{t,\,(p+1)}^{n-1,\,n-1}|| \leq \frac{1}{\sqrt{1+\varepsilon}}\,||\widetilde{\xi}_{t,\,(p)}^{n-1,\,n-1}||, \quad t\in\Delta,\ p\in\mathbb{N}.$$

Letting p run through $0, 1, \dots, p-1$, these inequalities add up to (69). $\square$

We now come to the key task of proving Lemma 4.7.

Proof of Lemma 4.7. Recall the notation $\widetilde{\xi}_{t,\,(0)}^{n-1,\,n-1} := \xi_t^{n-1,\,n-1}$, $\Omega_{t,\,(0)}^{n-1,\,n-1} := \Omega_t^{n-1,\,n-1}$ and $\nu_{t,\,(0)}^{n-1,\,n-1} := \nu_t^{n-1,\,n-1}$. Set $\varpi_{t,\,(p)} := \partial_t(\widetilde{\xi}_{t,\,(p)}^{n-1,\,n-1} + \nu_{t,\,(p)}^{n-1,\,n-1})$, the right-hand term of equations (67). The minimal L^2-norm solutions of equations (67) are explicitly given by the formulae :

$$\Omega_{t,\,(p)}^{n-1,\,n-1} = \Delta_t^{'-1}\partial_t^\star\varpi_{t,\,(p)} \quad \text{and} \quad \widetilde{\xi}_{t,\,(p+1)} = \Delta_t^{-1}d_t^\star\varpi_{t,\,(p)}, \qquad t\in\Delta,\ p\in\mathbb{N}. \tag{70}$$

Now it is easily seen that, for any ∂_t-exact $(r,\,s)$-form u on X_t, one has

$$||\Delta_t^{'-1}\partial_t^\star u|| = ||\Delta^{'-\frac{1}{2}}u||. \tag{71}$$

Indeed, if $(e_j^{r,\,s})_{j\in\mathbb{N}}$ is an orthonormal basis of $C_{r,\,s}^\infty(X_t,\,\mathbb{C})$ consisting of eigenvectors of Δ_t' such that $\Delta_t' e_j^{r,\,s} = \lambda_j\, e_j^{r,\,s}$ and if u splits as $u = \sum\limits_{j\in J_u} c_j\, e_j^{r,\,s}$ with $c_j\in\mathbb{C}$, then $e_j^{r,\,s}$ is ∂_t-exact for every $j\in J_u$ and

$$\Delta_t^{'-1}\partial_t^\star u = \sum_{j\in J_u}\frac{c_j}{\sqrt{\lambda_j}}\,e_j^{r-1,\,s},$$

where $(e_j^{r-1,\,s})_{j\in J_u}$ is an orthonormal subset of $C_{r-1,\,s}^\infty(X_t,\,\mathbb{C})$ consisting of eigenvectors of Δ_t' corresponding to the same eigenvalues as before : $\Delta_t' e_j^{r-1,\,s} = \lambda_j\, e_j^{r-1,\,s}$. This is because

$$\partial^\star \ :\ \mathrm{Im}\,(\partial : C_{r-1,\,s}^\infty \to C_{r,\,s}^\infty) \longrightarrow \mathrm{Im}\,(\partial^\star : C_{r,\,s}^\infty \to C_{r-1,\,s}^\infty)$$

is an angle-preserving isomorphism that maps any ∂-exact Δ'-eigenvector of type $(r,\,s)$ to a Δ'-eigenvector of type $(r-1,\,s)$ having the same eigenvalue λ and an L^2-norm multiplied by $\sqrt{\lambda}$. (We have suppressed indices t to ease the notation). A further application of $\Delta^{'-1}$ introduces divisions by the eigenvalues λ_j, hence the overall effect of applying $\Delta^{'-1}\partial^\star$ to u consists in multiplying the coefficients c_j by $\sqrt{\lambda_j}/\lambda_j = 1/\sqrt{\lambda_j}$ and replacing the orthonormal set of $(r,\,s)$-forms $\{e_j^{r,\,s},\, j\in J_u\}$ with an orthonormal set of $(r-1,\,s)$-forms $\{e_j^{r-1,\,s},\, j\in J_u\}$.

On the other hand, $\Delta^{'-\frac{1}{2}}u = \sum\limits_{j\in J_u}\frac{c_j}{\sqrt{\lambda_j}}\,e_j^{r,\,s}$. Hence, we see that

$$||\Delta_t^{'-1}\partial_t^\star u||^2 = ||\Delta'^{-\frac{1}{2}}u||^2 = \sum_{j\in J_u}\frac{|c_j|^2}{\lambda_j}.$$

Similarly, for any d-exact k-form u on X_t, one has

$$||\Delta_t^{-1}d_t^\star u|| = ||\Delta^{-\frac{1}{2}}u||. \tag{72}$$

Thus, in the light of (70),(71) and (72) with $u = \varpi_{t,(p)}$, the proof of Lemma 4.7 reduces to proving that

$$||\Delta_t^{-\frac{1}{2}}\varpi_{t,(p)}|| \leq \frac{1}{\sqrt{1+\varepsilon}}\,||\Delta_t^{'-\frac{1}{2}}\varpi_{t,(p)}||, \quad t\in\Delta, \;\; p\in\mathbb{N}. \tag{73}$$

We are thus led to compare the Laplacians Δ'_t and Δ_t for $t\in\Delta$. We begin by noticing that for any pure-type (say $(r,\,s)$) form u on some X_t that is not Δ''_t-harmonic, we have :

$$\langle\langle\Delta_t u,\, u\rangle\rangle > \langle\langle\Delta'_t u,\, u\rangle\rangle. \tag{74}$$

Indeed, by compactness of X_t, we get :

$$\begin{aligned}\langle\langle\Delta_t u,\, u\rangle\rangle &= ||d\,u||^2 + ||d_t^\star\,u||^2 \\ &= ||\partial_t u||^2 + ||\bar\partial_t u||^2 + ||\partial_t^\star u||^2 + ||\bar\partial_t^\star u||^2 \\ &> ||\partial_t u||^2 + ||\partial_t^\star u||^2 = \langle\langle\Delta'_t u,\, u\rangle\rangle.\end{aligned} \tag{75}$$

The equality between the top two lines follows from $d\,u = \partial_t u + \bar\partial_t u$ and the forms $\partial_t u$ and $\bar\partial_t u$ being orthogonal as pure-type forms of distinct types $(r+1,\,s)$ and respectively $(r,\,s+1)$. Thus, $||d\,u||^2 = ||\partial_t u||^2 + ||\bar\partial_t u||^2$ and the adjoints satisfy the analogous identity $||d_t^\star\,u||^2 = ||\partial_t^\star u||^2 + ||\bar\partial_t^\star u||^2$ for the same reasons. The strict inequality between the bottom two lines follows from the assumption that u is not Δ''_t-harmonic which amounts to $\bar\partial_t u$ and $\bar\partial_t^\star u$ not vanishing simultaneously. (Indeed, $\langle\langle\Delta''_t u,\, u\rangle\rangle = ||\bar\partial_t u||^2 + ||\bar\partial_t^\star u||^2$.)

Now, if u varies in a finite dimensional subspace $E_0 \subset C^\infty_{r,\,s}(X_0,\,\mathbb{C})$ of J_0-$(r,\,s)$-forms such that $E_0\cap\ker\Delta''_0 = \{0\}$, one can find a constant $\varepsilon>0$ such that inequality (74), when applied to Δ_0 and Δ'_0, strengthens to

$$\langle\langle\Delta_0 u,\, u\rangle\rangle \geq (1+\varepsilon)\,\langle\langle\Delta'_0 u,\, u\rangle\rangle, \quad \text{for all} \quad u\in E_0. \tag{76}$$

This is clear since, by (74), such an $\varepsilon>0$ can be found for every form u as above and the same ε can be kept for all forms on the complex line $\mathbb{C}\,u\subset E_0$. If there are only finitely many directions in E_0, the minimum of the finitely many corresponding constants can be chosen as the new $\varepsilon>0$.

Now $(\Delta_t)_{t\in\Delta}$ and $(\Delta'_t)_{t\in\Delta}$ are C^∞ families of operators since they are defined by metrics $(\gamma_t)_{t\in\Delta}$ that vary in a C^∞ way with $t\in\Delta$ (up to $t=0$). As a result, if a family of $(r,\,s)$-forms $(u_t)_{t\in\Delta}$ varies in a C^∞ way with t in a C^∞ vector subbundle of finite rank $E_t\subset C^\infty_{r,\,s}(X_t,\,\mathbb{C})$ such that $E_t\cap\ker\Delta''_t = \{0\}$

for all $t \in \Delta$, one can find a constant $\varepsilon > 0$ independent of $t \in \Delta$ (after possibly lowering the previous $\varepsilon > 0$ found for $t = 0$ in (76)) such that

$$\langle\langle \Delta_t u_t,\, u_t\rangle\rangle \geq (1+\varepsilon)\,\langle\langle \Delta'_t u_t,\, u_t\rangle\rangle, \quad \text{for all} \quad u_t \in E_t \quad \text{and all} \quad t \in \Delta, \tag{77}$$

after possibly shrinking the base Δ about 0. This amounts to

$$\Delta_t \geq (1+\varepsilon)\,\Delta'_t \quad \text{on} \quad E_t \subset C^\infty_{r,\,s}(X_t,\, \mathbb{C}), \quad t \in \Delta. \tag{78}$$

Now (75) and (77) show that such an ε must satisfy

$$0 < \varepsilon \leq \frac{\langle\langle \Delta''_t u_t,\, u_t\rangle\rangle}{\langle\langle \Delta'_t u_t,\, u_t\rangle\rangle}, \quad u_t \in E_t \setminus \ker \Delta'_t, \quad t \in \Delta. \tag{79}$$

By the choice $(\star)(a)$ of $\eta_{t,\,(p)} = \partial_t \nu^{n-1,\,n-1}_{t,\,(p)} = \bar\partial_t \vartheta^{n,\,n-2}_{t,\,(p)}$, the form $\varpi_{t,\,(p)} = \partial_t \widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)} + \eta_{t,\,(p)}$ is $\bar\partial_t$-exact for all $t \neq 0$, hence $\varpi_{t,\,(p)}$ is orthogonal to $\ker \Delta''_t$ for all $t \neq 0$. When $t = 0$, the form $\varpi_{0,\,(p)} = \partial_0 \widetilde{\xi}^{n-1,\,n-1}_{0,\,(p)} + \eta_{0,\,(p)}$ is not Δ''_0-harmonic (after possibly adjusting $\eta_{t,\,(p)}$ by a small factor $\delta'_p > 0$ independent of $t \in \Delta$) thanks to the choice of $\eta_{0,\,(p)}$ as a nonzero $\bar\partial_0$-exact form. It is in order to force this property that the forms $\eta_{t,\,(p)}$ were introduced and the construction of $\xi^{n-1,\,n-1}_{t,\,(p)}$ in Lemma 4.5 was altered to that of $\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)}$. Thus $\varpi_{t,\,(p)} \in E_{t,\,(p)}$ for all $t \in \Delta$ and some finite rank vector subbundle $E_{t,\,(p)} \subset C^\infty_{n,\,n-1}(X_t,\, \mathbb{C})$ satisfying $E_{t,\,(p)} \cap \ker \Delta''_t = \{0\}$ for all $t \in \Delta$. Moreover, in terms of $\varpi_{t,\,(p)}$, condition (c) of $(\star_p)$ (see choice of $\eta_{t,\,(p)}$) translates to the following conditions analogous to (80) for all $t \in \Delta$ and all $p \in \mathbb{N}$:

$$0 < \varepsilon_0 \leq \frac{\langle\langle \Delta''_t \varpi_{t,\,(p)},\, \varpi_{t,\,(p)}\rangle\rangle}{\langle\langle \Delta'_t \varpi_{t,\,(p)},\, \varpi_{t,\,(p)}\rangle\rangle}, \tag{80}$$

for an $\varepsilon_0 > 0$ independent of both $t \in \Delta$ and $p \in \mathbb{N}$. Set $\varepsilon = \varepsilon_0$ and get :

$$\langle\langle \Delta_t \varpi_{t,\,(p)},\, \varpi_{t,\,(p)}\rangle\rangle \geq (1+\varepsilon)\,\langle\langle \Delta'_t \varpi_{t,\,(p)},\, \varpi_{t,\,(p)}\rangle\rangle, \quad t \in \Delta, \;\; p \in \mathbb{N}. \tag{81}$$

If the operators Δ_t and Δ'_t commuted, the desired inequality (73) would follow by inverting the above inequalities. Although Δ_t and Δ'_t would commute if they were calculated with respect to a Kähler metric, commutation does not hold in general. Recall that the metric γ_t used here is only Gauduchon, but not necessarily Kähler. Therefore, rather than using general arguments, we shall obtain inequality (73) from (81) by means of specific considerations.

It suffices to treat the case $p = 0$ as the case of an arbitrary $p \in \mathbb{N}$ is similar. Set $\varpi_t := \varpi_{t,\,(0)}$ for all $t \in \Delta$. Consider the following decompositions :

$$\varpi_t = \sum_{j \in J} c_j(t)\, e^{n,\,n-1}_j(t) = \sum_{k \in K} d_k(t)\, f^{(2n-1)}_k(t), \quad t \in \Delta, \tag{82}$$

of ϖ_t with respect to orthonormal families $(e_j^{n,\,n-1}(t))_j$ and $(f_k^{(2n-1)}(t))_k$ of eigenvectors for Δ'_t and respectively Δ_t :

$$\Delta'_t e_j^{n,\,n-1}(t) = \lambda_j(t)\, t' e_j^{n,\,n-1}(t) \quad \text{and} \quad \Delta_t f_k^{(2n-1)}(t) = \mu_k(t)\, f_k^{(2n-1)}(t).$$

The index sets J and K are finite. As ϖ_t is of pure J_t-type $(n,\, n-1)$, the mutual orthogonality of the forms $f_k^{(2n-1)}(t)$ and of their respective Δ_t-eigenspaces implies that each $f_k^{(2n-1)}(t)$ is of pure J_t-type $(n,\, n-1)$. We clearly have

$$||\varpi_t||^2 = \sum_{j\in J} |c_j(t)|^2 = \sum_{k\in K} |d_k(t)|^2, \quad t \in \Delta. \tag{83}$$

On the other hand, we have $\Delta_t^{-1}\varpi_t = \sum\limits_{k\in K} \dfrac{d_k(t)}{\mu_k(t)}\, f_k^{(2n-1)}(t)$, hence

$$\langle\langle \Delta_t^{-1}(\varpi_t),\, \varpi_t\rangle\rangle = \sum_{k\in K} \frac{|d_k(t)|^2}{\mu_k(t)}, \quad t \in \Delta. \tag{84}$$

Similarly $\Delta_t^{'-1}(\varpi_t) = \sum\limits_{j\in J} \dfrac{c_j(t)}{\lambda_j(t)}\, e_j^{n,\,n-1}(t)$, hence

$$\langle\langle \Delta_t^{'-1}(\varpi_t),\, \varpi_t\rangle\rangle = \sum_{j\in J} \frac{|c_j(t)|^2}{\lambda_j(t)}, \quad t \in \Delta. \tag{85}$$

Pick any $k \in K$ and let $E_{\Delta_t}^{(2n-1)}(\mu_k(t))$ denote the Δ_t-eigenspace of $(2n-1)$-forms with eigenvalue $\mu_k(t)$. Consider the orthogonal decomposition

$$E_{\Delta_t}^{(2n-1)}(\mu_k(t)) \ni f_k^{(2n-1)}(t) = \sum_{l\in J_k(t)} u_{k,\,l}(t), \quad t \in \Delta, \tag{86}$$

where $u_{k,\,l}(t) \in E_{\Delta'_t}^{n,\,n-1}(\lambda_l(t))$ is the orthogonal projection of $f_k^{(2n-1)}(t)$ onto the Δ'_t-eigenspace $E_{\Delta'_t}^{n,\,n-1}(\lambda_l(t))$ with eigenvalue $\lambda_l(t)$ and $J_k(t) \subset J$ is the subset of indices $l \in J$ such that $u_{k,\,l}(t) \neq 0$. By orthogonality, we have

$$||f_k^{(2n-1)}(t)||^2 = 1 = \sum_{l\in J_k(t)} ||u_{k,\,l}(t)||^2. \tag{87}$$

Notice that $u_{k,\,l}(t) \in E_{\Delta_t}^{(2n-1)}(\mu_k(t))$ (and implicitly $u_{k,\,l}(t) \in E_{\Delta'_t}^{n,\,n-1}(\lambda_l(t)) \cap E_{\Delta_t}^{(2n-1)}(\mu_k(t))$) for all $l \in J_k(t)$. Indeed, if $u_{k,\,l_0}(t) \notin E_{\Delta_t}^{(2n-1)}(\mu_k(t))$ for some $l_0 \in J_k(t)$, the mutual orthogonality of the forms $u_{k,\,l}(t) \in E_{\Delta'_t}^{n,\,n-1}(\lambda_l(t))$ and of the spaces $E_{\Delta'_t}^{n,\,n-1}(\lambda_l(t))$ would make it impossible for $f_k^{(2n-1)}(t) = \sum\limits_{l\in J_k(t)} u_{k,\,l}(t)$ to belong to $E_{\Delta_t}^{(2n-1)}(\mu_k(t))$, a contradiction. Let

$$E'_k(t) \subset \bigoplus_{l\in J_k(t)} E^{n,\,n-1}_{\Delta'_t}(\lambda_l(t))$$

be the largest subspace onto which at least one of the forms $u_{k,\,l}(t)$, $l \in J_k(t)$, has a non-trivial orthogonal projection. Then each such orthogonal projection of every $u_{k,\,l}(t)$ still belongs to $E^{(2n-1)}_{\Delta_t}(\mu_k(t))$, hence

$$E'_k(t) \subset E^{(2n-1)}_{\Delta_t}(\mu_k(t)) \cap \bigoplus_{l\in J_k(t)} E^{n,\,n-1}_{\Delta'_t}(\lambda_l(t))$$

and $E'_k(t)$ is stable under both operators Δ_t and Δ'_t. This means that, taking restrictions to $E'_k(t)$, we get :

$$\Delta_t,\ \Delta'_t \ : \ E'_k(t) \to E'_k(t), \quad t \in \Delta. \tag{88}$$

Now, on the one hand, the restriction of Δ_t to $E'_k(t)$ has $\mu_k(t)$ as its unique eigenvalue, while the restriction of Δ'_t to $E'_k(t)$ has eigenvalues $\lambda_l(t)$, $l \in J_k(t)$. On the other hand, $E'_k(t)$ is orthogonal to $\ker \Delta''_t$ and (78) shows that these restrictions satisfy $\Delta_t \geq (1+\varepsilon)\,\Delta'_t$ for all $t \in \Delta$. Then the min-max principle gives

$$\mu_k(t) \geq (1+\varepsilon)\,\lambda_l(t), \quad \text{for all} \quad l \in J_k(t) \quad \text{and all} \quad t \in \Delta. \tag{89}$$

Now, in view of (86), for all $t \in \Delta$ we get the estimate :

$$\begin{aligned}
\langle\langle \Delta_t^{-1} f_k^{(2n-1)}(t),\, f_k^{(2n-1)}(t)\rangle\rangle &= \frac{1}{\mu_k(t)}\,||f_k^{(2n-1)}(t)||^2 = \frac{1}{\mu_k(t)} \\
&\leq \frac{1}{1+\varepsilon}\sum_{l\in J_k(t)} \frac{||u_{k,\,l}(t)||^2}{\lambda_l(t)} \\
&= \frac{1}{1-\varepsilon}\,\langle\langle \Delta_t^{'-1} f_k^{(2n-1)}(t),\, f_k^{(2n-1)}(t)\rangle\rangle.
\end{aligned} \tag{90}$$

As $k \in K$ has been chosen arbitrarily, this gives for all $t \in \Delta$:

$$\langle\langle \Delta_t^{-1}(\varpi_t),\, \varpi_t\rangle\rangle \leq \frac{1}{1+\varepsilon}\,\langle\langle \Delta_t^{'-1}(\varpi_t),\, \varpi_t\rangle\rangle, \tag{91}$$

an inequality that is equivalent to (73) by self-adjointness of $\Delta_t^{-\frac{1}{2}}$ and $\Delta_t^{'-\frac{1}{2}}$. The proof of Lemma 4.7 is complete. $\square$

End of proof of Proposition 4.1. By Corollary 4.8, the L^2-norm $||\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)}||$ can be made arbitrarily small, uniformly with respect to $t \in \Delta$, for $p \gg 1$ sufficiently large. Implicitly, so can the L^2-norm $||\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)} + \overline{\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)}}||$. Hence, after possibly shrinking Δ about 0, the strict positivity of $\gamma_t^{n-1,\,n-1}$ forces

the real $J_t-(n-1,\, n-1)$-form $\gamma_t^{n-1,\,n-1}-\widetilde{\xi}_{t,\,(p)}^{n-1,\,n-1}-\overline{\widetilde{\xi}_{t,\,(p)}^{n-1,\,n-1}}$ to be $\partial_t\bar{\partial}_t$-cohomologous to a positive definite $J_t-(n-1,\, n-1)$-form for all $t\in\Delta$ if $p\gg 1$ is sufficiently large. The last form can be written as $\rho_{t,\,(p)}^{n-1}>0$, where $\rho_{t,\,(p)}>0$ is a $\mathbb{C}^\infty$ form of J_t-type $(1,\,1)$ for every $t\in\Delta$, by Michelsohn's procedure [Mic83, p. 279-280] of extracting the $(n-1)^{st}$ root of any positive-definite $(n-1,\, n-1)$-form. Thus we get :

$$\rho_{t,\,(p)}^{n-1}-\left(\gamma_t^{n-1}-\widetilde{\xi}_{t,\,(p)}^{n-1,\,n-1}-\overline{\widetilde{\xi}_{t,\,(p)}^{n-1,\,n-1}}\right)\in \mathrm{Im}\,(\partial_t\bar{\partial}_t),\quad t\in\Delta,\;\; p\gg 1.$$

In particular,

$$\partial_t\rho_{t,\,(p)}^{n-1}=\partial_t\left(\gamma_t^{n-1}-\widetilde{\xi}_{t,\,(p)}^{n-1,\,n-1}-\overline{\widetilde{\xi}_{t,\,(p)}^{n-1,\,n-1}}\right),\quad t\in\Delta,\;\; p\gg 1.$$

By conclusion (66) of Lemma 4.6, $\partial_t\rho_{t,\,(p)}^{n-1}$ is $\bar{\partial}_t$-exact for all $t\in\Delta$. Thus, $(\rho_{t,\,(p)})_{t\in\Delta}$ is a family of *strongly Gauduchon* metrics on the fibres $(X_t)_{t\in\Delta}$ if $p\gg 1$. It modifies the original family $(\gamma_t)_{t\in\Delta}$. In particular, choosing any large $p\gg 1$, $\rho_{0,\,(p)}$ is a *strongly Gauduchon* metric on X_0. The proof of Proposition 4.1 is complete. $\square$

As explained earlier, Proposition 4.1 combined with Proposition 3.5 proved in the previous section proves Theorem 1.1.

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